80 Quantum Curvature as Key to the Quantum Universe

80 量子曲率: 量子宇宙的关键

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Contents

目录

Abstract

摘要

Curvature is a key notion in general relativity, characterizing the local physical properties of spacetime. By contrast, the concept of curvature has received scant attention in nonperturbative quantum gravity. One may even wonder whether in a Planckian regime, meaningful notions of (quantum) curvature exist at all. Remarkably, recent work in quantum gravity using causal dynamical triangulations (CDT) has demonstrated both the existence and usefulness of a new notion of quantum Ricci curvature (QRC), which relies neither on smooth structures nor on tensor calculus. This chapter recalls some classical notions related to curvature and parallel transport, as well as previous unsuccessful attempts to construct quantum curvature observables based on deficit angles and Wilson loops. It introduces the quasi-local QRC on piecewise flat triangulations and describes its behavior in a purely classical setting, its use in quantum observables, and its currently known results in (C)DT quantum gravity in two and four dimensions. The QRC opens the door to a range of interesting physical observables that were previously out of reach and will help to bridge the gap between the nonperturbative quantum theory and gravitational phenomena at lower energies.

曲率是广义相对论中的核心概念，用于描述时空的局部物理性质。与之相反，曲率的概念在非微扰量子引力中一直极少受到关注。人们甚至不禁疑问: 在普朗克尺度下，(量子) 曲率究竟是否存在有意义的定义。值得注意的是，近期借助因果动态三角剖分 (CDT) 开展的量子引力研究，已经证明了量子里奇曲率 (QRC) 这一新概念的存在性与实用性，该概念既不依赖光滑结构，也不依赖张量演算。本章回顾了与曲率、平行输运相关的经典概念，以及此前基于亏缺角和威尔逊圈构造量子曲率可观测量的失败尝试；介绍了分段平坦三角剖分上的准局域量子里奇曲率，描述了它在纯经典环境下的性质、它在量子可观测量中的应用，以及目前二维、四维 (因果) 动态三角剖分量子引力中已得到的相关结果。量子里奇曲率为一系列此前无法触及的有趣物理可观测量打开了大门，将有助于缩小非微扰量子理论与低能引力现象之间的鸿沟。

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Keywords

关键词

Quantum gravity - Causal dynamical triangulations - Dynamical triangulations - Quantum curvature - Nonperturbative - Lattice gravity - Ricci curvature

量子引力——因果动力学三角剖分——动力学三角剖分——量子曲率——非微扰——格点引力——里奇曲率

Introduction

引言

Anyone who starts learning about the mathematical structure of general relativity quickly understands that the notion of curvature plays a pivotal role in the theory. Curvature, which describes how a spacetime deviates locally from being flat, appears on the left-hand side of the Einstein equations

任何开始学习广义相对论数学结构的人都会很快认识到，曲率概念在该理论中发挥着核心作用。曲率描述时空局部如何偏离平直性，它出现在爱因斯坦方程的左侧

$$R_{\mu\upsilon}(x) - \frac{1}{2}g_{\mu\upsilon}(x)R(x) = 8\pi G_N T_{\mu\upsilon}(x) \tag{1}$$

in the form of the Ricci tensor $R_{\mu\upsilon}(x)$ and the Ricci scalar $R(x)$, both obtained by contractions of the full Riemann curvature tensor $R^{\kappa}{}_{\lambda\mu\nu}(x)$. The Einstein equations relate the curvature to the distribution of matter and energy, as captured by the energy-momentum tensor $T_{\mu\nu}$ on the right-hand side of Eq. (1). The fact that we model curved spacetimes in general relativity by differentiable manifolds M equipped with a metric tensor $g_{\mu\nu}$ goes back to the foundational work of Riemann in the mid-nineteenth century. As described in his famous habilitation thesis [1], he was guided by "experience" - i.e., his physical, Newtonian intuition - to introduce the infinitesimal line element

其形式为里奇张量 $R_{\mu\upsilon}(x)$ 和里奇标量 $R(x)$ ，二者均由完整黎曼曲率张量 $R^{\kappa}{}_{\lambda\mu\nu}(x)$ 缩并得到。爱因斯坦方程将曲率与物质和能量的分布联系起来，后者由式 (1) 右侧的能量动量张量 $T_{\mu\nu}$ 描述。广义相对论中用带有度规张量 $g_{\mu\nu}$ 的微分流形 M 建模弯曲时空，可以追溯到黎曼在 19 世纪中期的基础性工作。正如他在著名的任教资格论文 [1] 中所述，他受" 经验"——即他的牛顿物理学直觉——引导，引入了无穷小线元

$$ds^2 = g_{\mu\nu}(x)\,dx^{\mu}dx^{\nu}. \tag{2}$$

The immense power of the formalism of Riemannian geometry, when generalized to four-dimensional Lorentzian (pseudo-Riemannian) spacetimes, is reflected in the spectacular successes of general relativity, which has given us black holes, dynamical universes, gravitational waves, and more. Remarkably from a modern point of view, Riemann already anticipated that "[his] considerations may not apply in the immeasurably small" and may have to be revisited in the light of new physical observations.

当推广到四维洛伦兹 (伪黎曼) 时空后，黎曼几何形式体系的强大威力，在广义相对论的惊人成就中得到了充分体现: 它为我们带来了黑洞、动态宇宙、引力波等等诸多成果。从现代视角来看非常值得注意的是，黎曼早已预见到"[他的] 结论可能不适用于无穷小尺度"，并可能需要根据新的物理观测重新审视。

This is precisely the situation we find ourselves in when trying to understand and describe the microscopic structure of spacetime in a nonperturbative quantum theory of gravity. Although we cannot currently observe physics at the Planck scale directly, our understanding of the nature of quantum fields at very short distances, acquired since Riemann's days, leads us to believe that the metric fields of the classical formulation may not be adequate to describe the properties of quantum spacetime in the vicinity of the Planck scale. Generalizing or altogether abandoning the metric (or an equivalent field, like the vierbein (tetrad) $e^{\mu}_{a}(x)$ of a first-order formulation of gravity) in these circumstances raises the question of what happens to curvature.

这正是我们尝试在非微扰量子引力理论中理解和描述时空微观结构时所处的境况。尽管我们目前无法直接观测普朗克尺度的物理，但自黎曼时代以来我们对极短距离量子场性质的认知让我们认为，经典表述中的度规场可能不足以描述普朗克尺度附近量子时空的性质。在这种情况下，推广或是完全放弃度规 (或是等价场，比如一阶引力表述中的 vierbein(tetrad) $e_a^\mu(x)$)，自然会引出曲率会如何变化的问题。

In this work, we will examine the quest for curvature in a relatively conservative quantum scenario, where a metric structure is still present but is not smooth. The concrete setting we will be working with is lattice gravity in terms of causal dynamical triangulations (CDT), where the gravitational path integral is defined nonperturbatively as a continuum limit of a sum over piecewise flat triangulated spaces [2, 3] . However, given the simplicity of the ingredients, many of our considerations and constructions are likely applicable in other regularized or discretized formulations of quantum gravity too. The centerpiece of our exposition is the quantum Ricci curvature, a recently introduced notion of curvature [4, 5], whose novelty is twofold: it has been used to construct quantum curvature observables that have been shown to be well defined and finite in the quantum theory, and it is a quantum implementation of genuine Ricci curvature, which captures directional curvature information beyond what is contained in the Ricci scalar.

在本文中，我们将在一个相对保守的量子情景中探究曲率: 该情景下仍然保留度规结构，但度规不再光滑。我们采用的具体框架是基于因果动态三角剖分 (CDT) 的格点引力，其中引力路径积分被非微扰地定义为分段平直三角剖分空间求和的连续极限 [2, 3] 。但由于我们采用的基础要素非常简洁，我们的许多讨论和构造很可能也适用于其他正则化或离散化的量子引力表述。本文的核心是量子里奇曲率，这是一个近年提出的曲率概念 [4,5]，它的创新体现在两方面: 我们利用它构造的量子曲率可观测量，已经被证实在量子理论中是良定义且有限的；同时它是真正里奇曲率的量子实现，能够捕获里奇标量无法包含的方向相关曲率信息。

The remainder of this chapter is structured as follows. To set the stage, section "Curvature: Classical Foundations" contains a brief review of classical Riemannian curvature, including the related notions of parallel transport and holonomy. Motivated by quantum considerations, we describe in section "Taking a Quantum Perspective" the concept of the deficit angle, a standard way of quantifying the curvature of piecewise flat spaces, like those of Regge calculus or dynamical triangulations. We discuss the shortcomings of this prescription, and that of the closely related gravitational Wilson loop, in the nonperturbative quantum theory. In section "Quantum Ricci Curvature", we define the quantum Ricci curvature (QRC) and its implementation. We summarize what is known about its behavior on classical smooth spaces and a variety of piecewise flat spaces. We introduce the associated diffeomorphism-invariant curvature profile and discuss the averaging properties of the QRC. Section "Quantum Ricci Curvature: Quantum Applications" is devoted to proper quantum applications of the QRC, in 2D toy models of Lorentzian (CDT) and Euclidean (DT) quantum gravity, and in the full 4D theory defined in terms of CDT, which has produced important additional evidence for the de Sitter nature of its emergent geometry. The presence of a well-defined notion of quantum curvature in the nonperturbative quantum theory opens new channels of investigation into the nature of quantum spacetime at the Planck scale, as we will outline briefly in section "Summary and Outlook". It also brings us a step closer to (semi-)classical treatments of gravity, including in the very early universe, to which we ultimately want to connect our Planckian findings.

本章剩余部分结构安排如下。为铺垫背景，“曲率: 经典基础”一节简要回顾经典黎曼曲率，包含相关的平行移动与和乐概念。受量子考量的启发，我们在“量子视角”一节描述亏缺角的概念——这是量化分段平坦空间 (例如里奇微积分或动力三角化中的空间) 曲率的标准方法。我们讨论了该方案以及与之密切相关的引力威尔逊圈在非微扰量子理论中的缺陷。在“量子里奇曲率”一节，我们定义了量子里奇曲率 (QRC) 并说明其实现方式，总结了目前已知的它在经典光滑空间和多种分段平坦空间上的行为，引入了相关的微分同胚不变曲率分布，并讨论了量子里奇曲率的平均性质。“量子里奇曲率: 量子应用”一节专门介绍量子里奇曲率在合适的量子场景中的应用，包括洛伦兹型 (CDT) 和欧几里得型 (DT) 量子引力的二维玩具模型，以及由 CDT 定义的完整四维理论——该理论已经为其涌现几何的德西特本质提供了重要的额外证据。非微扰量子理论中存在明确定义的量子曲率，为探究普朗克尺度量子时空的本质开辟了新的研究路径，我们会在“总结与展望”一节对此做简要概述。这也让我们向着 (半) 经典引力处理 (含极早期宇宙的引力处理) 更近了一步，我们最终希望将普朗克尺度的研究结论与该方向联系起来。

Curvature: Classical Foundations

曲率: 经典基础

Before discussing possible generalizations, let us recall some aspects of classical curvature to provide background and context. We will only consider intrinsic curvature, which is independent of any embedding of the space or spacetime under consideration. One way to operationally detect curvature is by following initially parallel geodesics. On a flat space, they will remain parallel, while in the presence of curvature, they can move toward or away from each other (see Fig. 1). The equation of geodesic deviation describes how the relative acceleration of neighboring geodesics is related to the Riemann tensor $R^{\kappa}{}_{\lambda\mu\nu}(x)$; see, e.g.,[6].

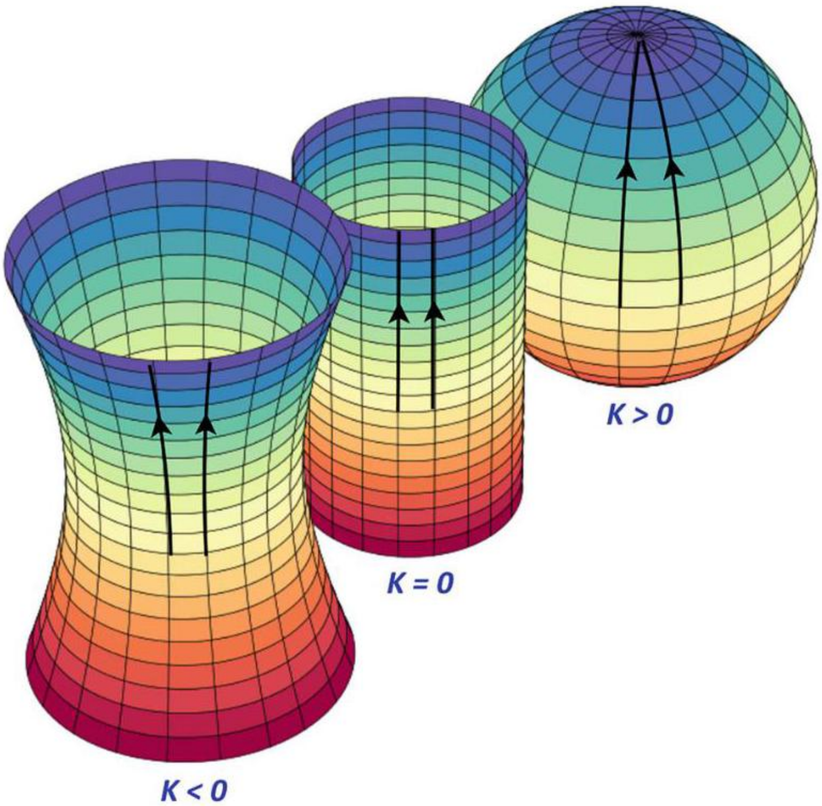
在讨论可能的推广之前，我们先回顾经典曲率的若干内容，以提供背景与语境。我们仅讨论内蕴曲率，它不依赖于所研究空间或时空的任何嵌入。可操作探测曲率的一种方法是追踪初始平行的测地线: 在平直空间中，它们会保持平行；而存在曲率时，它们会相互靠近或远离 (见图 1)。测地线偏离方程描述了相邻测地线的相对加速度如何与黎曼张量 $R^{\kappa}{}_{\lambda\mu\nu}(x)$ 相关联，参见例如文献 [6]。

The larger the dimension D of a space, the more complicated are its curvature properties: the Riemann tensor, which depends on the metric $g_{\mu\nu}(x)$ and its first and second derivatives, has $D^2\left(D^2-1\right)/12$ independent components, that is,1,6, and 20 components in $D = 2, 3$, and 4, respectively. Another potential source of complication, especially from the viewpoint of the quantum theory, is the fact that the standard classical description of geometry and gravity has a large degree of redundancy, associated with the freedom to choose any coordinate system $\{x^{\mu}\}$ on the underlying spacetime. The explicit functional form of the metric $g_{\mu\nu}(x)$ and other tensorial quantities depends on this choice and will transform nontrivially when performing a coordinate change. However, the physical, geometric properties of a spacetime are independent of such a coordinate choice and invariant under a change of coordinates. Like in a gauge field theory, in general relativity, one therefore should distinguish carefully between ”physics” and ”gauge,” that is, genuine coordinate- (or diffeomorphism-)invariant behavior and unphysical ”coordinate effects”. (Whenever talking about coordinate invariance, we mean invariance under the diffeomorphism group $\mathrm{Diff}(M)$ as an active group of point transformations of the manifold M .)

空间的维度 D 越大，其曲率性质就越复杂: 依赖于度量 $g_{\mu\nu}(x)$ 及其一阶、二阶导数的黎曼张量共有 $D^2(D^2-1)/12$ 个独立分量，分别对应维度 $D=2,3$ 和 4 维空间的 1 个、6 个和 20 个分量。另一个复杂来源，尤其从量子理论视角来看，是标准的经典几何与引力描述存在大量冗余，这源于基础时空上可任意选择坐标系 $\{x^\mu\}$ 的自由度。度量 $g_{\mu\nu}(x)$ 和其他张量量的显式函数形式依赖于该选择，并且在坐标变换下会发生非平凡变换。但时空的物理几何性质独立于坐标选择，在坐标变换下保持不变。因此和规范场论类似，在广义相对论中我们必须仔细区分“物理”与“规范”，也就是真正的坐标 (或微分同胚) 不变行为，以及非物理的“坐标效应”。(我们提及坐标不变性时，指的是在微分同胚群 Diff(M) 下的不变性，该群是流形 M 上主动的点变换群。)

Fig. 1 The behavior of initially parallel geodesics depends on the Gaussian curvature K of the underlying two-dimensional manifold

图 1 初始平行测地线的行为由基础二维流形的高斯曲率 K 决定



Taking the arbitrariness of the coordinate choice into account, curvature appears to be a more physical notion than the metric, in the sense that the former cannot be ”transformed away”: by going to the coordinate system of a freely falling observer, a so-called local inertial frame, the metric $g_{\mu\nu}$ is transformed locally to the metric $\eta_{\mu\nu}$ of flat Minkowski space, with vanishing first derivatives, but its curvature (second derivatives) in general cannot be made to vanish. Similarly, local scalars under coordinate transformations are curvature invariants like the Ricci scalar $R(x)$ or the Kretschmann scalar $R^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu}(x)$. It suggests that in the context of a generalized, nonclassical geometric setting, some form of quantum curvature may be a more physical notion than that of a ”quantum metric $\hat{g}_{\mu\nu}$.” This is to some extent borne out by the nonperturbative formulation based on CDT, which does not suffer from coordinate redundancies, and allows us to introduce the quantum Ricci curvature (see sections "Quantum Ricci Curvature" and "Quantum Ricci Curvature: Quantum Applications").

考虑到坐标选择的任意性，曲率相比度规是一个更具物理性的概念，因为曲率无法被“变换消除”：通过转换到自由下落观测者的坐标系 (即所谓局部惯性系)，度规 $g_{\mu\nu}$ 会在局域变换为平坦闵氏空间的度规 $\eta_{\mu\nu}$ ，此时其一阶导数为零，但它的曲率 (二阶导数) 通常无法整体变为零。同理，坐标变换下的局部标量就是曲率不变量，例如里奇标量 $R(x)$ 或克雷奇曼标量 $R^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu}(x)$ 。这说明，在推广的非经典几何框架中，某种形式的量子曲率相比“量子度规 $\hat{g}_{\mu\nu}$ ”是更具物理性的概念。这一点在一定程度上得到了基于 CDT 的非微扰表述的验证: 该表述不存在坐标冗余，还允许我们引入量子里奇曲率 (参见“量子里奇曲率”与“量子里奇曲率: 量子应用”两节)。

An alternative to geodesic deviation to extract the curvature of a metric manifold $(M, g_{\mu\nu})$ is to consider the holonomies of infinitesimal closed curves in M . Given a parametrized path $\gamma(s) : [0,1] \mapsto M$, with $\gamma(0) = x_0$ and $\gamma(1) = x_1$, the holonomy $U(\gamma; x_0, x_1)$ is the path-ordered exponential of the metric-compatible connection $\Gamma^{\mu}_{\kappa\upsilon}(x)$,

提取度量流形 $(M, g_{\mu\nu})$ 曲率的另一种替代方法是测地偏差法，即考虑 M 中无穷小闭合曲线和乐。给定参数化路径 $\gamma(s) : [0,1] \mapsto M$ ， 满足 $\gamma(0) = x_0$ 和 $\gamma(1) = x_1$ ， 和乐 $U(\gamma; x_0, x_1)$ 是度量相容联络 $\Gamma^{\mu}_{\kappa\upsilon}(x)$ 的路径序指数,

$$U(\gamma; x_0, x_1) = \mathrm{Pe} - \int_0^1 ds \Gamma_\kappa(\gamma(s))\dot{\gamma}^\kappa(s), \; (\Gamma_\kappa)^\mu{}_\upsilon \; := \; \Gamma^\mu_{\kappa\upsilon}, \tag{3}$$

which is independent of the parametrization of γ . The path-ordering prescription ("P") in Eq. (3) is needed because of the non-abelian character of the connection Γ . The holonomy describes the parallel transport of a vector $V^\mu(x_0)$ from the point x_0 to the point x_1 along the path γ (Fig. 2, left) according to

它与 γ 的参数化无关。式 (3) 中的路径排序规则 (标记为"P") 是必须的，因为联络 Γ 具有非阿贝尔性质。和乐描述了向量 $V^\mu(x_0)$ 按照下述规则沿路径 γ 从点 x_0 到点 x_1 的平行移动 (图 2 左):

$$V^\mu(x_1) = U(\gamma; x_0, x_1)^\mu{}_\nu V^\nu(x_0). \tag{4}$$

In a construction that may be familiar from gauge field theory (with a $su(N)$ - valued gauge connection $A_\mu(x)$ instead of the Levi-Civita connection $\Gamma_\mu(x)$), one can extract curvature information by considering the holonomy of an infinitesimal closed square loop $\gamma_{[\mu\upsilon]}$ of side length ε in the $\mu - \upsilon$ -plane with base point x , and expanding it in powers of ε ,

在规范场论中 (我们用 $su(N)$ 值规范联络 $A_\mu(x)$ 替代列维-奇维塔联络 $\Gamma_\mu(x)$)， 大家可能对这个构造很熟悉: 我们可以通过考虑基点为 x 、位于 $\mu - \upsilon$ 平面内、边长为 ε 的无穷小闭合方形环路 $\gamma_{[\mu\upsilon]}$ 的和乐来提取曲率信息，并将其按 ε 的幂次展开:

$$U(\gamma_{[\mu\upsilon]}; x) = \mathrm{P}\exp\oint_{\gamma[\mu\upsilon]}(-\Gamma) = \mathbb{1} + \varepsilon^2 R_{\mu\upsilon}(x) + \mathcal{O}(\varepsilon^3) = \mathrm{e}^{\varepsilon^2 R_{\mu\upsilon}} + \mathcal{O}(\varepsilon^3), \tag{5}$$

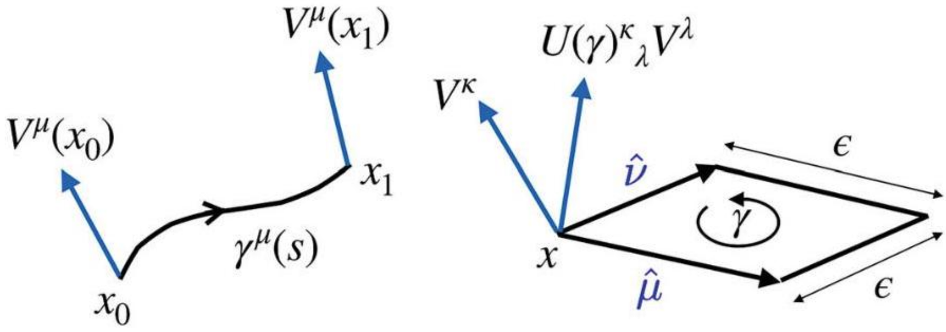


Fig. 2 Left: Parallel transport of a vector V along an open path γ . Right: Parallel transport of a vector V around an infinitesimal square path γ spanned by two unit vectors in the μ - and υ -directions

图 2 左: 向量 V 沿开路径 γ 的平行移动。右: 向量 V 绕无穷小方形路径 γ 的平行移动，该路径由 μ 方向和 υ 方向的两个单位向量张成

where $(R_{\mu\upsilon})^\kappa{}_\lambda := R^\kappa{}_{\lambda\mu\upsilon}$. Depending on whether the metric is Riemannian or pseudo-Riemannian, the matrix $R_{\mu\nu}(x)$ for fixed μ, υ generates an infinitesimal $SO(D)$ - or $SO(1, D-1)$ -rotation in the tangent space T_xM at the base point x of the loop $\gamma_{[\mu\nu]}$ (Fig. 2, right). Relation (5) illustrates the origin of the complexity of curvature, in the sense that the number of components of the Riemann tensor grows very quickly with increasing D : it is due to the various D -dimensional rotations associated with every choice of a local bi-vector spanning an infinitesimal square loop $\gamma_{[\mu\upsilon]}$ around which a vector can be parallel-transported. Lastly, note that the group-valued rotation matrix (5) still transforms nontrivially under a coordinate transformation. To reduce this coordinate dependence, one can take its trace to obtain a "gravitational Wilson loop" $W_{\gamma_{[\mu\upsilon]}}[\Gamma] = \mathrm{Tr}\, U(\gamma_{[\mu\upsilon]}; x)$, which is a scalar with respect to linear transformations of the tangent space at x .

其中 $(R_{\mu\upsilon})^\kappa{}_\lambda := R^\kappa{}_{\lambda\mu\upsilon}$。根据度量是黎曼度量还是伪黎曼度量，固定 μ, υ 时的矩阵 $R_{\mu\nu}(x)$ 会在环路 $\gamma_{[\mu\nu]}$ 基点 x 的切空间 T_xM 中生成无穷小 $SO(D)$ 旋转或 $SO(1, D-1)$ 旋转 (图 2 右)。关系式 (5) 说明了曲率复杂性的来源: 黎曼张量的分量数随 D 增大而快速增长，这是因为，每选择一个张成无穷小方形环路 $\gamma_{[\mu\upsilon]}$ (可绕它平行移动向量) 的局部双矢量，都会对应不同的 D 维旋转。最后注意，式 (5) 中群值旋转矩阵在坐标变换下仍会发生非平凡变换。为弱化这种坐标依赖性，可以对它取迹，得到" 引力威尔逊圈" $W_{\gamma_{[\mu\upsilon]}}[\Gamma] = \mathrm{Tr}\, U(\gamma_{[\mu\upsilon]}; x)$ ， 它相对于 x 处切空间的线性变换是标量。

Taking a Quantum Perspective

采用量子视角

The classical curvature constructions reviewed in the previous section rely on the presence of a differentiable manifold and the associated tensor calculus. In addition, when solving the Einstein equations (1), one usually requires the metric to be at least twice differentiable. As already mentioned in the introduction, none of these features may be realized in a nonperturbative regime of quantum gravity. The construction of quantum observables - a key aim of any quantum theory - must therefore rely on less regular ingredients. It raises the question of whether meaningful notions of curvature, or quantum implementations thereof, can be defined in such a context at all. This issue cannot be answered straightforwardly, because it goes beyond the realm of standard perturbative quantum field theory, where local fields and operators are generically singular, but the smooth nature of the underlying manifold is not usually questioned.

上一节回顾的经典曲率构造依赖于可微流形和相关张量演算的存在。此外，求解爱因斯坦方程 (1) 时，通常要求度规至少二阶可微。正如引言中已经提到的，在量子引力的非微扰 regime 下，这些性质都可能不复存在。因此，构造量子可观测量——任何量子理论的核心目标——必须依赖正则性更弱的结构。这就引出一个问题: 在这样的背景下，究竟能否定义有意义的曲率概念，或是曲率的量子实现形式。这个问题无法直接回答，因为它超出了标准微扰量子场论的范畴；在标准微扰量子场论中，局部场和算符通常存在奇异性，但一般不会质疑底流形的光滑性。

It is tempting to argue on physical grounds that a notion like curvature, which plays such a key role in classical general relativity, ought to have some correlate in the quantum theory. Otherwise, one would have to envisage a mechanism by which curvature merely "emerges" macroscopically from some (sub-)Planckian spacetime substrate. Given the highly complex nature of curvature in four dimensions, this is a challenging task. Ultimately, it is a question about the true, microscopic nature of spacetime, one of the core issues a theory of quantum gravity is expected to settle.

人们很容易从物理角度提出: 曲率在经典广义相对论中发挥着核心作用，这样一个概念理应在量子理论中存在对应。否则就必须设想一种机制，让曲率仅在宏观尺度从 (亚) 普朗克时空基底中“涌现”出来。鉴于曲率在四维空间中高度复杂的性质，这是一项极具挑战性的任务。归根结底，这是一个关于时空真实微观本质的问题，也是量子引力理论需要解决的核心问题之一。

There are clearly also technical issues that will need to be addressed. Even on a smooth spacetime background, curvature is a second-order differential operator, which in the quantum theory will generally require regularization and renormalization. This situation is unlikely to improve for less regular backgrounds. An example of what one may expect are the typical spacetime configurations that contribute to nonperturbative gravitational path integrals formulated in terms of dynamical triangulations (see Fig. 14 for a 2D example), which in a continuum limit can be thought of as higher-dimensional analogues of the nowhere differentiable paths of the path-integral quantization of a nonrelativistic particle (see, e.g., [7] for a discussion of the latter).

显然还有诸多技术问题需要解决。即使在光滑时空背景上，曲率也是二阶微分算符，在量子理论中通常需要正则化和重整化。对于正则性更弱的背景，情况不太可能变得更好。一个典型的预期例子是: 以动力学三角剖分表述的非微扰引力路径积分中，贡献占比最大的典型时空构型 (二维例子见图 14)，在连续极限下，可以看作非相对论粒子路径积分量子化中处处不可微路径的高维类比 (例如参见 [7] 对后者的讨论)。

Further difficulties may be present if one adopts graph-like or discrete models of spacetime, as advocated in some approaches to quantum gravity, for example, the causal set approach [8]. Nevertheless, as we will show in sections "Quantum Ricci Curvature" and "Quantum Ricci Curvature: Quantum Applications" by explicit construction, a notion of Ricci curvature can be defined and implemented successfully in specific nonperturbative quantum-gravitational settings, thereby answering the general existence question in the affirmative. Before delving into the details, we will in the following two subsections discuss previous attempts at defining curvature observables in quantum gravity, involving the notion of a deficit angle, and the gravitational Wilson loop mentioned above. Finally, let us mention that there is ongoing research in mathematics on developing generalized notions of curvature, which can be applied on nonsmooth metric spaces, in both Riemannian and Lorentzian signature; see, e.g., [9,10].

如果采用类图或离散时空模型 (部分量子引力研究方法支持这类模型，例如因果集方法 [8])，还会面临更多困难。尽管如此，正如我们将在“量子里奇曲率”和“量子里奇曲率: 量子应用”两节中通过显式构造展示的，里奇曲率概念可以在特定非微扰量子引力背景下成功定义和实现，从而对这个一般性的存在问题给出了肯定回答。在深入细节之前，我们将在接下来的两个小节中讨论量子引力中定义曲率可观测量的过往尝试，包括涉及亏缺角的方案和前文提到的引力威尔逊圈。最后，我们要说明，数学领域目前仍在研究发展适用于非光滑度量空间的广义曲率概念，涵盖黎曼和洛伦兹两种号差；例如参见 [9,10]。

Curvature as Deficit Angles

以亏缺角表示曲率

A time-honored way to describe the spacetime dynamics of general relativity in an approximate manner, without ever introducing coordinates, is Regge calculus [11]. Its key idea is to approximate metric manifolds $(M, g_{\mu\nu})$ by piecewise flat, triangulated spaces, so-called simplicial manifolds. In this setting, a triangulated spacetime T is characterized uniquely by the squared edge lengths $\{\ell_i^2, i = 1, 2, ...\}$ of its elementary building blocks, the D -simplices (D -dimensional generalizations of triangles), and by gluing or connectivity data, which specify how the simplices are assembled into D -dimensional spaces by identifying them pairwise along matching (D-1)-dimensional faces (Fig. 3, left). (The use of squared edge lengths is important in Lorentzian signature, to distinguish between time-, space-, and lightlike edges. Note that the flat, interior geometry of a D -simplex is uniquely determined by its (squared) edge lengths.). This method was originally devised to numerically solve the classical Einstein equations in the absence of symmetries [12,13]. However, as has become apparent over time, the true power of using piecewise flat spaces lies in providing a regularization for the curved spacetimes $[g] \in \mathcal{G}$ that are summed (or integrated) over in a nonperturbative version of the gravitational path integral, aka the ”sum over histories”

里奇微积分是一种由来已久的无需引入坐标即可近似描述广义相对论时空动力学的方法 [11]。其核心思想是用分片平坦的三角剖分空间 (即单纯流形) 来近似度量流形 $(M, g_{\mu\nu})$ 。在该框架下，三角剖分时空 T 可由两个要素唯一确定: 一是其基本构造单元 D 单形 (三角形的 D 维推广) 的边长平方 $\{\ell_i^2, i = 1, 2, ...\}$ ，二是粘合/连通数据，该数据规定了如何通过沿匹配的 (D-1) 维面对两两匹配单形进行等同，将它们组装为 D 维空间 (图 3 左)。(在洛伦兹号差中使用边长平方十分重要，可区分类时、类空和类光边。需要注意的是，D 单形平坦的内蕴几何由其 (平方) 边长唯一确定。) 该方法最初是为了在无对称性的情况下数值求解经典爱因斯坦方程而设计的 [12,13]。但随着时间推移人们逐渐发现，使用分片平坦空间的真正作用在于，它为非微扰引力路径积分 (即“历史求和”) 中需要对其求和 (或积分) 的弯曲时空 $[g] \in \mathcal{G}$ 提供了正则化方案。

$$Z = \int_{\mathcal{G}} \mathcal{D}[g]\, e^{iS[g]}, \quad \text{with } S[g] = \frac{1}{16\pi G_N} \int_M d^4x\, (R - 2\Lambda)\,. \tag{6}$$

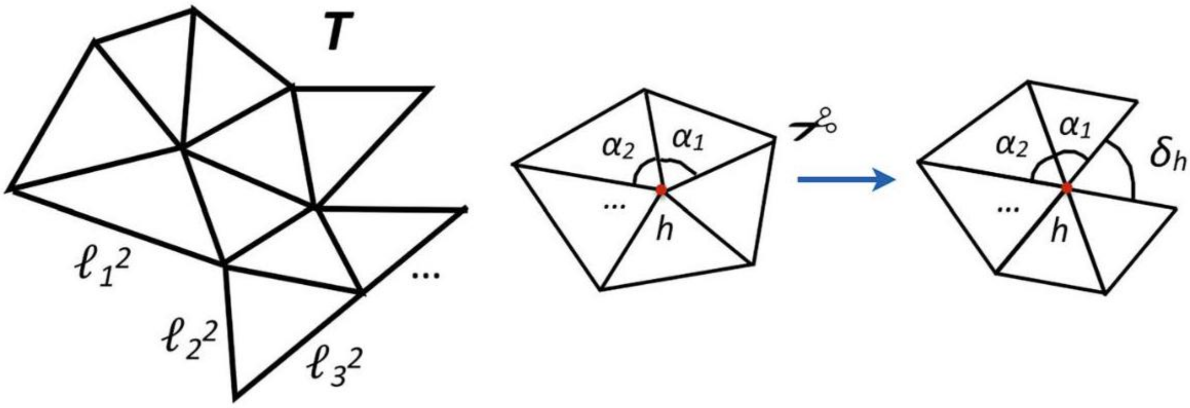


Fig. 3 Left: Two-dimensional flat triangles with edges of squared length $\ell_1^2, \ell_2^2, \ell_3^2, \ldots$, are glued together pairwise, yielding a finite triangulation T with the topology of a disc. Right: Each of the triangles meeting at the vertex h contributes an angle α_i . Cutting the triangulation open as indicated and putting it on a flat surface reveals a deficit angle δ_h

图 3 左: 两个带边长平方 $\ell_1^2, \ell_2^2, \ell_3^2, \ldots$ 边的二维平坦三角形两两粘合，得到一个具有圆盘拓扑的有限三角剖分 T 。右: 交汇于顶点 h 的每个三角形贡献一个角度 α_i 。按图示方式切开三角剖分并将其铺在平面上，即可得到亏缺角 δ_h

In (6), $[g]$ labels a geometry, i.e., an equivalence class of metrics under the action of the diffeomorphism group $\mathrm{Diff}(M)$; $S[g]$ is the gravitational Einstein-Hilbert action, including a cosmological term; G_N is Newton' s constant; and Λ is the cosmological constant. In CDT, the formal expression (6) is substituted by a concrete prescription, which defines the nonperturbative path integral Z as a continuum limit of a regularized sum over causal triangulations,

在式 (6) 中，$[g]$ 标记一个几何，即微分同胚群作用下的度量等价类；$\mathrm{Diff}(M)$; $S[g]$ 是引力爱因斯坦-希尔伯特作用量，包含宇宙学项；G_N 是牛顿引力常数；Λ 是宇宙学常数。在因果动态三角剖分 (CDT) 中，形式表达式 (6) 被具体规则替代，该规则将非微扰路径积分 Z 定义为对因果三角剖分正则化求和的连续极限。

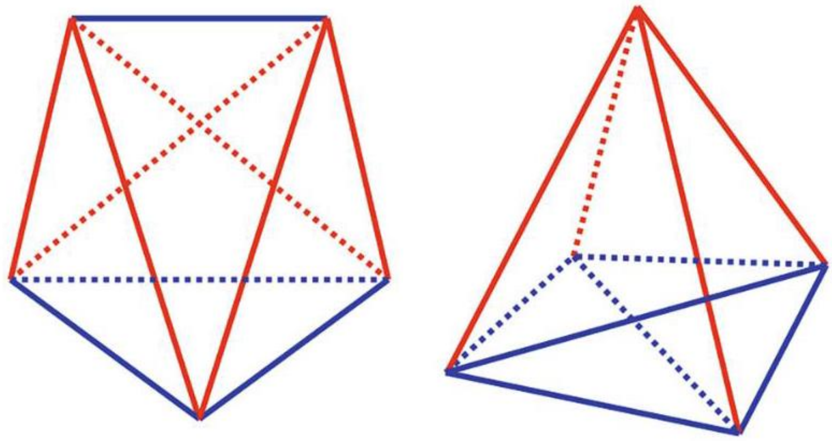
$$Z = \int \mathcal{D}[g]\, \mathrm{e}^{iS[g]} \overset{CDT}{\to} Z = \lim_{N_4 \to \infty} \sum_{\text{causal } T} \frac{1}{C(T)} \mathrm{e}^{iS[T]}, \tag{7}$$

where $C(T)$ is the number of elements in the automorphism group of the triangulation $T, S[T]$ is the simplicial implementation of the Einstein-Hilbert action [11], and N_4 is the upper cutoff on the number of four-dimensional simplicial building blocks contained in T . By construction, a causal triangulation T contributing to (7) contains only two types of edges, spacelike edges of squared length $\ell_s^2 = a^2$ and timelike edges of squared length $\ell_t^2 = -\alpha a^2$, for some fixed $\alpha > 0$. There are (up to time orientation) two geometrically distinct flat building blocks in 4D CDT (see Fig. 4), the $(3, 2)$ -simplex and the $(4, 1)$ -simplex. The notation (m, n) refers to the "stacked" structure of CDT geometries, where each vertex lies in a three-dimensional spatial triangulation of fixed integer proper (lattice) time. A (3,2)-simplex has three vertices with time label t and two vertices with time label $t + 1$ (or the other way round), while a $(4, 1)$ -simplex has four vertices with time label t and one vertex with time label $t + 1$ (or the other way round). One key advantage of representing geometries by piecewise flat spaces in the implementation by CDT is the absence of (coordinate) redundancies in Z , thereby avoiding any technical complications related to gauge-fixing (see [2, 3, 14] for further technical details and references).

其中 $C(T)$ 是三角剖分自同构群的元素个数, $T, S[T]$ 是爱因斯坦-希尔伯特作用量的单纯形实现 [11], N_4 是 T 中包含的四维单纯形构建块数量的上限。根据构造，对 (7) 有贡献的因果三角剖分 T 仅包含两类边: 固定 $\alpha > 0$ 下，平方长度为 $\ell_s^2 = a^2$ 的类空边和平方长度为 $\ell_t^2 = -\alpha a^2$ 的类时边。在四维因果动态三角剖分 (CDT) 中存在 (相差时间定向后) 两种几何上不同的平直构建块 (见图 4)，即 (3, 2) 单纯形和 (4, 1) 单纯形。记号 (m, n) 指代 CDT 几何的“分层”结构: 每个顶点都位于固定整数固有时 (格点) 的三维空间三角剖分上。(3,2) 单纯形有三个顶点的时间标记为 t ，两个顶点的时间标记为 $t+1$ (反之亦可)；而 (4, 1) 单纯形有四个顶点的时间标记为 t ，一个顶点的时间标记为 $t+1$ (反之亦可)。CDT 框架下用分段平直空间表示几何的一个核心优势是，Z 中不存在 (坐标) 冗余，因此避免了所有与规范固定相关的技术难题 (更多技术细节和参考文献参见 [2, 3, 14])。

Fig. 4 The two building blocks of 4D CDT are four-dimensional Minkowskian simplices with spacelike edges of squared length a^2 (blue) and timelike edges of squared length $-\alpha a^2$ (red): the (3, 2)-simplex (left) and the (4, 1) -simplex (right)

图 4 四维 CDT 的两种构建块，为四维闵氏单纯形，平方长度为 a^2 的类空边 (蓝色) 和平方长度为 $-\alpha a^2$ 的类时边 (红色): 左为 (3, 2) 单纯形，右为 (4, 1) 单纯形



(3,2) (4,1)

Important for our purposes is how curvature is encoded in triangulations of Regge-type. Like Regge calculus itself, this works in the same way for any dimension D and metric signature, where appropriate attention should be paid to the nature of Minkowskian angles when the signature is indefinite [15]. Looking at a simple example of a triangulation, like that depicted in Fig. 3, left, for $D = 2$ and positive definite signature, the presence of curvature may not be obvious, since a planar drawing cannot in general represent the edge lengths faithfully. Nevertheless, curvature is present at any vertex of such a triangulation where the angles of the triangles sharing it do not sum up to 2π , leading to a nonvanishing positive or negative deficit angle. For general dimension D , the deficit angle δ_h associated with a $(D-2)$ -dimensional subsimplex ("hinge") of a triangulation is defined by

我们研究的关键是曲率如何编码在里奇型三角剖分中。与里奇微积分本身一样，这套表示方法对任意维数 D 和任意度规号差都成立，仅需在号差不定时适当注意闵氏角的性质 [15]。我们来看一个简单的三角剖分例子，即图 3 左图所示、$D = 2$ 、正定号差的情况，曲率的存在可能并不直观，因为平面绘图通常无法忠实表示边长。尽管如此，这种三角剖分的任意顶点处都存在曲率: 共享该顶点的所有三角形内角之和不等于 2π ，从而产生一个非零的正或负亏缺角。对于任意维数 D ，三角剖分中与 $(D-2)$ 维子单纯形 (“铰链”) 关联的亏缺角 δ_h 定义为

$$\delta_h = 2\pi - \sum_{i|\Delta_i \supset h} \alpha_i \tag{8}$$

where the sum is over all dihedral angles α_i at the hinge h of the D -simplices Δ_i sharing h . The situation in $D = 2$ and for a positive deficit angle δ_h is illustrated in Fig. 3, right. In two dimensions, the deficit angle (8) is a direct measure of the Gaussian curvature associated in a distributional manner with the vertex h . This can be seen by parallel-transporting a vector along a closed loop γ in the triangulation, which is well defined as long as γ does not cross or touch the vertex h (or any other vertex). Let us for simplicity assume that the loop lies inside the disc of triangles sharing h and is neither self-intersecting nor self-touching. Then, whenever γ does not contain h in its interior, the holonomy is the two-dimensional unit matrix, $U(\gamma; x) = \mathbb{1}$, for any base point x , reflecting the flatness of the geometry enclosed by the loop. By contrast, if h lies inside γ , the holonomy will be a $SO(2)$ -rotation matrix corresponding to the angle $\delta \bmod 2\pi$, for a suitably chosen orientation of γ . More precisely, the dimensionless number δ_h represents Gaussian curvature (with dimension of inverse length-squared) integrated over a two-dimensional area element associated with the vertex h . A possible choice is given by the area of the polygon dual to h , whose corners are the barycenters of the triangles sharing h .

其中求和遍及共享枢轴 h 的所有 D -单形 Δ_i 在 h 处的所有二面角 α_i 。$D = 2$ 中正亏缺角 δ_h 的情形如图 3 右图所示。二维情况下，亏缺角 (8) 是分布在顶点 h 上的高斯曲率的直接度量。我们可以通过对三角剖分中的闭合环路 γ 上的矢量做平行输运看出这一点: 只要 γ 不穿过或接触顶点 h (或其他任意顶点)，平行输运就是良定义的。为简化起见，我们假设环路位于共享 h 的三角形构成的圆盘内部，且既不自交也不自切。那么，只要 γ 内部不包含 h ，对任意基点 x ，和乐就是二维单位矩阵 $U(\gamma; x) = \mathbb{1}$ ，这反映出环路包围的几何是平坦的。相反，如果 h 位于 γ 内部，当 γ 方向选取合适时，和乐就是对应角度 $\delta \bmod 2\pi$ 的 $SO(2)$ 旋转矩阵。更准确地说，无量纲量 δ_h 代表分布在与顶点 h 关联的二维面积元上的高斯曲率 (高斯曲率量纲为长度平方的倒数) 的积分。一种常用的选取是 h 对偶多边形的面积，该多边形的顶点是所有共享 h 的三角形的重心。

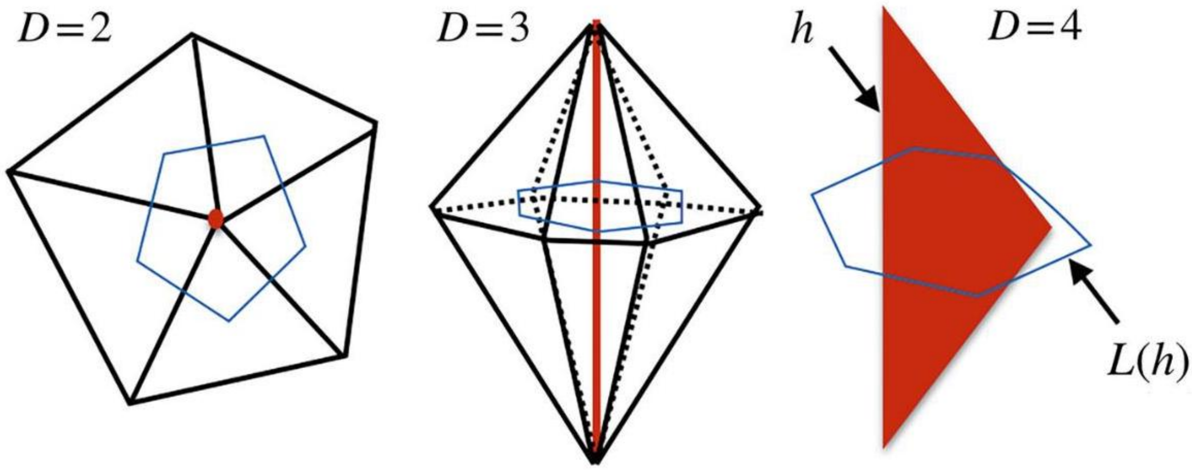


Fig. 5 Curvature in a D -dimensional simplicial manifold is concentrated at hinges h of dimension $D-2$: vertices in $D=2$, edges in $D=3$, and triangles in $D=4$ (in red). Its magnitude is given by the deficit angle δ_h at h , Eq. (8), and can be extracted by considering the holonomy of a small loop $L(h)$ around h (in blue)

图 5 D 维单纯形流形中的曲率集中于维度为 $D-2$ 的铰链 h 处: $D=2$ 中为顶点，$D=3$ 中为边，$D=4$ 中为三角形 (红色显示)。曲率的大小由 h 处的亏缺角 δ_h 给出，见式 (8)，可通过分析环绕 h 的小回路 $L(h)$ (蓝色显示) 的和乐得到

In an analogous manner, the curvature of a higher-dimensional simplicial manifold is associated with the deficit angles at its (D-2)-dimensional hinges h according to expression (8). The curvature at a given h is directly related to the holonomy of a small loop $L(h)$ encircling h in a two-dimensional area perpendicular to the hinge (see Fig. 5). An important quantity in gravity is the integrated scalar curvature, which is part of the Einstein-Hilbert action $S[g]$ in (6), and a particular instance of a diffeomorphism-invariant observable $\mathcal{O}$. Following Regge, its simplicial analogue in any dimension $D \geq 2$ is a weighted sum of the deficit angles of all hinges in the triangulation T ,

类似地，根据表达式 (8)，高维单纯形流形的曲率与 (D-2) 维枢轴 h 处的亏缺角相关联。给定枢轴 h 处的曲率，与垂直于该枢轴的二维平面内环绕 h 的小环路 $L(h)$ 的和乐直接相关。引力中的一个重要物理量是积分标量曲率，它是 (6) 式中爱因斯坦-希尔伯特作用量 $S[g]$ 的组成部分，也是微分同胚不变可观测量 $\mathcal{O}$ 的一个特例。按照 Regge 的理论，任意维度 $D \geq 2$ 下，它在单纯形引力中的对应形式是三角剖分 T 中所有枢轴亏缺角的加权和，

$$\frac{1}{2}\int_M d^D x \sqrt{|\det g|}\,{}^{(D)}R \overset{\text{Regge}}{\to} \sum_{h\in T} \mathrm{vol}(h)\,\delta_h, \tag{9}$$

where ${}^{(D)}R$ denotes the D -dimensional Ricci scalar and $\mathrm{vol}(h)$ the $(D-2)$ - dimensional volume of the hinge h . As already mentioned, Regge calculus is an approximation method and therefore subject to the usual discretization ambiguities, for example, how local volumes and curvatures are associated with particular subsimplices of a triangulation (see, e.g., [16]). It should not be mistaken as an exact representation of general relativity. In what way its expressions and equations converge to those of the classical theory has been investigated in some detail. However, we will not discuss these specifics here since they are not directly relevant for our quantum considerations below.

其中 $^{(D)}R$ 代表 D 维里奇标量，$\mathrm{vol}\,(h)$ 代表枢轴 h 的 $(D-2)$ 维体积。正如前文所述，Regge 微积分是一种近似方法，因此存在常规的离散化歧义，例如局部体积和曲率如何与三角剖分的特定子单形关联 (参见例如文献 [16])。它不能被误认为是广义相对论的精确表示。已有研究详细讨论过其表达式和方程收敛到经典理论对应形式的方式，但我们在此不讨论这些细节，因为它们和我们下文的量子讨论没有直接关联。

Deficit Angle Curvature in Dynamical Triangulations

动态三角剖分中的亏角曲率

In preparation for the quantum implementation of curvature, we will consider next the deficit angle prescription of the previous subsection for the special case of the triangulated spacetimes that appear in the gravitational path integral of CDT. From now on, our discussion will concentrate on spacetimes of positive definite signature, which is the case relevant for the analytically continued path integral of the CDT formulation. Recall that CDT quantum gravity possesses an explicit and well-defined ”Wick rotation,” given by the analytic continuation $\alpha \to -\alpha$ in the lower complex α -plane of the parameter α that characterized the length ℓ_t of timelike edges, as introduced below Eq. (7). This analytic continuation maps the regularized CDT version of the Lorentzian path integral (7) to a corresponding Euclidean path integral

为了准备曲率的量子实现，我们接下来将上一小节的亏角方案应用于 CDT 引力路径积分中出现的特殊情形——三角剖分时空。从现在开始, 我们的讨论将集中在正定号差的时空上, 这一情形与 CDT 框架下解析延拓后的路径积分相关。回顾可知，CDT 量子引力具有明确且定义良好的“威克转动”，由 (7) 式下方引入的解析延拓 $\alpha \to -\alpha$ 给出，该延拓在复参数 α 平面的下半部分进行，参数 α 刻画类时边的长度 ℓ_t 。该解析延拓将洛伦兹型路径积分 (7) 的正则化 CDT 版本映射为对应的欧几里得路径积分

$$Z^{\mathrm{eu}} = \lim_{N_4 \to \infty} \sum_{\mathrm{causal}\ T} \frac{1}{C\,(T)} \mathrm{e}^{-S^{\mathrm{eu}}[T]}, \tag{10}$$

where T labels Wick-rotated causal, piecewise flat triangulations and S^{eu} is the Euclidean Regge action. This analytical continuation is needed to explicitly evaluate the sum over geometries Z and the expectation values

其中 T 标记威克转动后的因果分片平坦三角剖分，S^{eu} 是欧几里得里奇作用量。这一解析延拓是显式计算几何求和 Z 与期望值得出的

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \sum_{\mathrm{causal}\ T} \frac{1}{C\,(T)} \mathcal{O}\,[T]\, \mathrm{e}^{-S[T]}, \tag{11}$$

of observables $\mathcal{O}$ [2, 3, 17] , where in (11) and in what follows, we have dropped the explicit superscript ”eu” for ”Euclidean”. (As was demonstrated first in two spacetime dimensions [18], working with an analytically continued Lorentzian CDT path integral yields continuum results inequivalent to those of Euclidean quantum gravity in terms of dynamical triangulations (DT), whose starting point is a sum over Riemannian spaces, and which does not possess a natural notion of Wick rotation.)

可观测量 $\mathcal{O}$ [2, 3, 17]，在 (11) 式及后续内容中，我们省略了表示“欧几里得”的显式上标“eu”。(正如文献 [18] 首先在二维时空证明的那样，使用解析延拓后的洛伦兹 CDT 路径积分得到的连续谱结果，与动态三角剖分 (DT) 框架下欧几里得量子引力的结果不等价；后者以黎曼空间求和为出发点，不存在自然的威克转动概念。)

As already mentioned, a distinguishing feature of the piecewise flat geometries contributing to the CDT path integral is the fact that their lattice edges come in two types of different lengths, depending on whether an edge was time- or spacelike before the Wick rotation. This is different from Euclidean DT, where there is only one possible edge length and all triangulations are therefore equilateral. Both situations can be viewed as particular instances of the simplicial manifolds considered in Regge calculus, which among other things allows us to use the Regge prescription (9) for the integrated scalar curvature. To simplify the analysis of this quantity, we will evaluate it on an equilateral triangulation; doing the same for a non-equilateral CDT configuration is slightly more involved, but does not alter the substance of the argument.

如前所述，对 CDT 路径积分有贡献的分片平坦几何的一个显著特征是: 格点边分为两种不同长度的类型，分类依据是威克转动前边是类时边还是类空边。这与欧几里得 DT 不同，欧几里得 DT 仅有一种可能的边长度，因此所有三角剖分都是等边的。两种情形都可以看作里奇微积分中单纯形流形的特殊实例，这允许我们对整体标量曲率使用里奇方案 (9)。为简化该量的分析，我们将在等边三角剖分上计算它；对非等边 CDT 构型做相同计算仅复杂度稍高，但不会改变论证的核心。

For an equilateral D -dimensional simplex, all dihedral angles, that is, all interior angles α_D between pairs of adjacent $(D-1)$ -dimensional faces, have the same value $\alpha_D = \arccos 1/D$. As a consequence, the deficit angle δ_h at a $(D-2)$ -dimensional hinge h in an equilateral simplicial manifold T can take a discrete set of values

对于等边 D 维单纯形，所有二面角 (即相邻 $(D-1)$ 维面之间的所有内角 α_D) 都取相同值 $\alpha_D = \arccos 1/D$ 。因此，等边单纯形流形 T 中 $(D-2)$ 维铰链 h 处的亏角 δ_h 只能取一组离散值

$$\delta_h = 2\pi - c_h \alpha_D, \tag{12}$$

where the coordination number $c_h = 3, 4, 5, \ldots$ is the number of D -simplices sharing h . (The fact that T is a simplicial manifold implies a lower bound of 3 for the number c_h of D -simplices allowed to meet at a hinge.) For the simplest nontrivial case $D = 2$, one obtains

其中配位数 $c_h = 3, 4, 5, \ldots$ 是共享 h 的 D 单纯形的数量。(T 是单纯形流形这一事实给出了铰链处允许交汇的 D 单纯形数量 c_h 的下界为 3。) 对于最简单的非平凡情形 $D = 2$ ，可得

$$\delta_h = 2\pi - c_h \pi/3 \ \ (\mathrm{D} = 2). \tag{13}$$

The two-dimensional case is special, in the sense that Regge's deficit angle prescription (9) for the total curvature, with δ_h given by Eq. (13), satisfies an exact identity. Recall that the integrated scalar curvature of a smooth geometry $(M, g_{\mu\nu})$ on a two-dimensional compact manifold M without boundary satisfies the so-called Gauss-Bonnet theorem

二维情形是特殊的: 总曲率的里奇亏角方案 (9) 在 δ_h 由 (13) 式给出时满足一个精确恒等式。回顾可知，无边界二维紧致流形 M 上光滑几何 $(M, g_{\mu\nu})$ 的整体标量曲率满足所谓的高斯-博内定理

$$\frac{1}{2}\int_M d^2x\sqrt{g}R = 2\pi\chi(M), \tag{14}$$

where $\chi(M)$ denotes the Euler characteristic of M , a topological invariant. Since for a vertex h we have $\mathrm{vol}(h) = 1$, the total deficit angle curvature of a two-dimensional equilateral triangulation is given by

其中 $\chi(M)$ 表示 M 的欧拉示性数，是一个拓扑不变量。由于对顶点 h 有 $\mathrm{vol}(h) = 1$ ，二维等边三角剖分的总亏角曲率可写为

$$\sum_{h\in T}\delta_h = \sum_{h\in T}(2\pi - c_h\pi/3) = 2\pi N_0 - \pi N_2 = 2\pi(N_0 - N_1 + N_2) = 2\pi\chi(M), \tag{15}$$

where N_0, N_1 , and N_2 are the numbers of vertices, edges, and triangles in T , respectively, and we have used various identities satisfied by these counting variables. It follows that the simplicial geometries, which we introduced as approximate representations of smooth curved spaces, satisfy the same Gauss-Bonnet formula, if their curvature is expressed in terms of deficit angles. As we will argue later, from the point of view of the nonperturbative quantum theory, this behavior appears to be a curiosity particular to two dimensions, which does not necessarily imply that the deficit angle curvature is distinguished or preferred on physical grounds.

其中 N_0, N_1 、N_2 分别是 T 中顶点、边和三角形的数量，我们此处用到了这些计数变量满足的多个恒等式。由此可得，我们引入用来近似表示光滑弯曲空间的单纯形几何，在曲率以亏缺角表示时，同样满足高斯-博内公式。我们后文会说明，从非微扰量子理论的角度来看，这一性质似乎只是二维特有的有趣结论，并不意味着亏缺角曲率在物理层面是特殊或优选的。

However, the main problem appears when we go to higher dimensions and consider the total deficit angle curvature, i.e., the expression on the right-hand side of (9) in the continuum limit of the (C)DT path integral, which is given by the combined limit $N_D \to \infty$ and $a \to 0$, while keeping the physical, dimensionful D -volume $V_D = a^D N_D$ fixed. The dimensionful lattice spacing a sets the scale of the edge or link length and plays the role of an ultraviolet length cutoff. To understand the behavior of the total curvature of a typical geometry in this limit, we re-express (9) in powers of N_D , disregarding constant terms, which yields

然而，当我们进入更高维、研究总亏缺角曲率时，就会出现核心问题: 也就是 (C)DT 路径积分连续统极限下式 (9) 右侧的表达式，该极限是在固定量纲物理 D 体积 $V_D = a^D N_D$ 的情况下，同时取 $N_D \to \infty$ 和 $a \to 0$ 的极限。量纲晶格间距 a 确定了边/链长度的尺度，充当紫外长度截断的角色。为了理解该极限下典型几何的总曲率行为，我们将式 (9) 按 N_D 的幂次重新展开，忽略常数项，得到

$$\sum_{h\in T}\mathrm{vol}(h)\,\delta_h \sim N_{D-2}\,\mathrm{vol}(h)\,\bar{\delta}_h \sim N_D\left(N_D^{-1/D}\right)^{D-2}\bar{\delta}_h = N_D^{2/D}\bar{\delta}_h, \tag{16}$$

where $\bar{\delta}_h$ denotes the average deficit angle and the second proportionality follows from $N_{D-2} \sim N_D$ and the fact that the hinge volume scales like a $(D-2)$ -volume. For $D = 2$, the final expression in (16) is $N_2\bar{\delta}_h = N_2 2\pi\chi/N_0 \sim$ const, a finite constant, but for $D = 4$, one obtains $N_4^{1/2}\bar{\delta}_h$, which in the limit $N_4 \to \infty$

diverges to infinity, because the average deficit angle is (numerically) observed to not scale to zero in this limit [19].

其中 $\overline{\delta}_h$ 表示平均亏缺角，第二个比例关系由 $N_{D-2} \sim N_D$ 以及枢轴体积按 $(D-2)$ 体积标度的性质得出。对于 $D = 2$ ，式 (16) 的最终结果为 $N_2\overline{\delta}_h = N_2 2\pi\chi/N_0 \sim$ 乘以常数，即一个有限常数；但对于 $D = 4$ ，可得 $N_4^{1/2}\overline{\delta}_h$ ，在 $N_4 \to \infty$ 极限下发散到无穷大，因为 (数值观测表明) 平均亏缺角在该极限下并不会标度到零 [19]。

From a quantum field-theoretic point of view, this divergence is not necessarily pathological, but means that the ”naïve” deficit angle curvature needs to be renormalized to become physically meaningful. However, there is currently no suggestion for how to do this. In the absence of a physically well-motivated renormalization prescription, we will define an alternative measure of curvature in section ”Quantum Ricci Curvature”. It not only turns out to be better behaved in the continuum limit but is a direction-dependent Ricci curvature, going beyond the scalar character of (9). Before introducing this so-called quantum Ricci curvature, we will briefly return to the gravitational Wilson loop as another potential measure of curvature, which is closely related to the deficit angle.

从量子场论的角度来看，这种发散不一定是病态的，它只说明“朴素”的亏缺角曲率需要重整化才能具有物理意义。但目前尚未有人提出可行的重整化方案。在缺乏物理动机充分的重整化规则的情况下，我们将在“量子里奇曲率”一节中定义一种替代的曲率度量。该度量不仅被证明在连续统极限下性质更好，而且是方向依赖的里奇曲率，超越了式 (9) 的标量特性。在引入这种所谓的量子里奇曲率之前，我们先简要讨论引力威尔逊圈，它是另一种潜在的曲率度量，与亏缺角密切相关。

Curvature from Wilson Loops

来自威尔逊回路的曲率

We saw in section ”Curvature: Classical Foundations” that the holonomy of a closed loop in a manifold $(M, g_{\mu\upsilon})$ contains information about its curvature, which appeared explicitly in the expansion (5) for the holonomy $U(\gamma_{[\mu\upsilon]}; x)$ of an infinitesimal square loop in the $\mu - \upsilon$ -plane. In search of curvature observables for the quantum theory, one may therefore consider the nonlocal gravitational Wilson loop

我们在“曲率: 经典基础”小节中已经看到，流形 $(M, g_{\mu\upsilon})$ 中闭合回路的和乐包含了其曲率的信息，这一点在 $\mu - \upsilon$ 平面内无穷小方形回路的和乐 $U(\gamma_{[\mu\upsilon]}; x)$ 展开式 (5) 中已经明确体现。因此，在寻找量子理论的曲率可观测量时，我们可以考虑非局域引力威尔逊回路

$$W_\gamma[\Gamma] = \mathrm{Tr}\, U(\gamma) \tag{17}$$

associated with a closed loop γ in M , which was already mentioned at the end of section ”Curvature: Classical Foundations”. Note that due to the cyclicity of the trace, W_γ no longer depends on the choice of the base point x of the loop γ . The first difficulty one encounters is that, unlike what happens in gauge field theory, the Wilson loop (17) of the Levi-Civita connection Γ is not an observable in pure gravity, for the same reason that the Ricci scalar $R(x)$ is not an observable: although both are scalar quantities with respect to the frame rotations induced by a diffeomorphism, such a diffeomorphism will in general move their arguments, that is, the point x or the loop γ . Neither the Ricci scalar at a point x nor the Wilson loop (17) of a loop γ are therefore diffeomorphism-invariant quantities.

关联到 M 中的闭合回路 γ ，这一点我们已经在“曲率: 经典基础”小节末尾提到过。注意由于迹的循环性质，W_γ 不再依赖于回路 γ 基点 x 的选择。我们遇到的第一个难点是，和规范场论中的情况不同，列维-奇维塔联络 Γ 的威尔逊回路 (17) 并不是纯引力中的可观测量，原因和里奇标量 $R(x)$ 不是可观测量的原因相同: 尽管二者相对于微分同胚诱导的标架旋转都是标量，但一般来说微分同胚会移动它们的自变量，也就是点 x 或回路 γ 。因此，点 x 处的里奇标量和回路 γ 的威尔逊回路 (17) 都不是微分同胚不变量。

While it is straightforward to construct a diffeomorphism-invariant observable from $R(x)$ by integrating it over the manifold M , as was done on the left-hand side of (9), integrating W_γ over all loops γ is neither practical nor interesting, since W_γ is itself already a nonlocal quantity depending in a complicated way on γ and Γ . Alternatively, one could try to average the Wilson loop over a class of loops that share certain invariant geometric features regarding their length and shape. How to do this in a background-independent manner is not an easy task. Moreover, it is not even clear in principle what type of curvature information could be retrieved in this way from Wilson loops of non-infinitesimal size, as we will explain further below.

虽然我们可以像 (9) 式左侧那样，通过将 $R(x)$ 在流形 M 上积分，轻易从 $R(x)$ 构造出微分同胚不变的可观测量，但将 W_γ 在所有回路 γ 上积分既不实用也没有意义，因为 W_γ 本身已经是一个非局域量，以复杂的方式依赖于 γ 和 Γ 。另一种方法是，我们可以尝试对满足长度和形状特定不变几何特征的一类回路，对威尔逊回路做平均。如何用背景无关的方式实现这一点并非易事。此外，正如我们在下文会进一步解释，原则上我们甚至不清楚这种方式能从非无穷小尺寸的威尔逊回路中提取出何种曲率信息。

On the plus side, holonomies and their associated Wilson loops are easily implemented and evaluated on the piecewise flat geometries of the CDT path integral [20], unlike on smooth curved manifolds, where they generically cannot be computed at all. A straightforward prescription on a four-dimensional triangulation is to use straight path segments between the centers of adjacent four-simplices and consider the holonomies of piecewise straight loops made from such segments. To compute the corresponding path-ordered exponentials, one must introduce a coordinate system in each simplex at an intermediate stage of the calculation, but the dependence on this choice drops out when taking the trace. Working with this particular class of closed curves does not represent a loss of generality, since a continuous deformation of a loop within the flat parts of a triangulation (i.e., avoiding its curvature singularities) leaves the associated Wilson loop invariant.

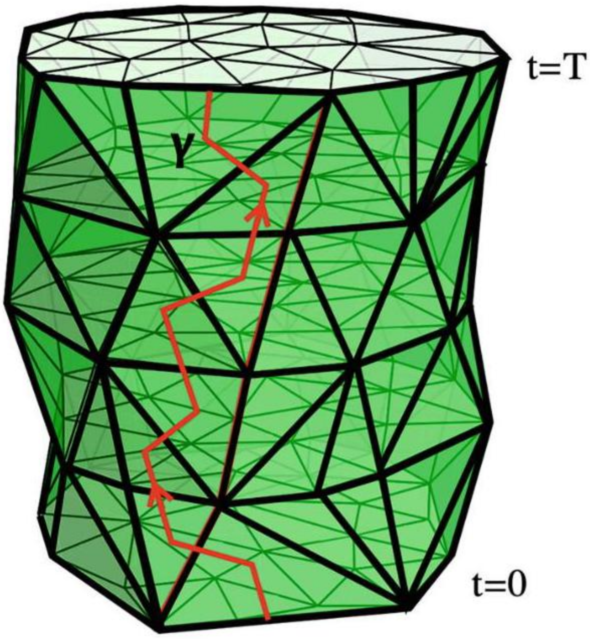
另一方面，和乐及其关联的威尔逊回路在 CDT 路径积分的分段平坦几何上很容易实现和计算 [20]，这和光滑弯曲流形不同——在光滑流形上一般根本无法计算它们。在四维三角剖分上有一种直接的方案: 使用相邻四单形中心之间的直线路段，考虑由这类路段构成的分段直回路的和乐。要计算对应的路径序指数，我们必须在计算的中间阶段给每个单形引入坐标系，但在取迹之后，对坐标系的依赖就会消失。使用这类特殊闭合曲线不会损失一般性，因为回路在三角剖分平坦部分 (即避开曲率奇点的部分) 内的连续形变不会改变关联的威尔逊回路。

The feasibility of implementing Wilson loops in four-dimensional CDT quantum gravity was demonstrated in [20], which considered the expectation value of a Wilson loop whose underlying loop coincides with the worldline of a massive particle, cyclically identified in time (Fig. 6). This is a well-defined observable, since the particle ”marks” the loop in a diffeomorphism-invariant way. From the Wilson loop measurements, one could extract the expectation value of the probability distribution $P(\theta_1, \theta_2)$ of two invariant angles $\theta_i \in [0, \pi]$ characterizing $SO(4)$ - elements up to conjugation and show that they are uniformly distributed with respect to the Haar measure on the group manifold of $SO(4)$. It implies that, at least within the measuring accuracy of [20], this particular Wilson loop observable does not carry any interesting curvature information (see also the remarks in [3], Section 7.2).

已有文献 [20] 论证了在四维 CDT 量子引力中实现威尔逊圈的可行性，该研究探讨了一个威尔逊圈的期望值: 其基础回路与时间上循环标识的有质量粒子世界线重合 (图 6)。这是一个定义明确的可观测量，因为粒子以微分同胚不变的方式“标记”了该回路。从威尔逊圈测量中，我们可以提取出刻画 $SO(4)$ 的两个不变角 $\theta_i \in [0, \pi]$ 的概率分布 $P(\theta_1, \theta_2)$ 的期望值——共轭等价类意义下的 $SO(4)$ 元素，并证明它们相对于 $SO(4)$ 群流形上的哈尔测度是均匀分布的。这意味着，至少在文献 [20] 的测量精度范围内，这个特殊的威尔逊圈可观测量不携带任何有价值的曲率信息 (也可见文献 [3] 第 7.2 节的评述)。

Fig. 6 Schematic representation of the worldline of a particle, moving forward in time t along a piecewise straight path in a CDT configuration, consisting of segments between centers of adjacent d -simplices. Cyclical identification of the initial and final time results in a closed loop γ whose corresponding Wilson loop W_γ , after summing over all loops of this type, is a quantum observable [20]

图 6 粒子世界线的示意图，该粒子沿时间向前运动 t ，在 CDT 构型中沿分段直路径运动，路径由相邻 d 单形中心之间的线段组成。将初始时间和终点时间循环标识后得到一个闭合回路 γ ，其对应的威尔逊环 W_γ 在对所有此类回路求和后是一个量子可观测量 [20]



Next, let us look at how a Wilson loop may in principle encode gauge-invariant information about the curvature of the underlying space. On a smooth four-dimensional Riemannian manifold, the gravitational Wilson loop of an infinitesimal square loop $\gamma_{[\mu v]}$ of geodesic edge length ε in the $\mu - v$ plane depends to lowest nontrivial order on the entries $R_{\cdot,\mu v}$ of the Riemann curvature tensor according to

接下来，我们来看威尔逊环原则上如何编码底层空间曲率的规范不变信息。在光滑四维黎曼流形上，无穷小方形回路 $\gamma_{[\mu\upsilon]}$ (测地线边长为 ε ，位于 $\mu - \upsilon$ 平面内) 的引力威尔逊环，其最低非平凡阶依赖于黎曼曲率张量的分量 $R_{\cdot,\mu\upsilon}$ ，关系如下

$$W_{\gamma[\mu\upsilon]} = 4 + \varepsilon^4 R^{\kappa}{}_{\lambda\mu\upsilon} R^{\lambda}{}_{\kappa\mu\upsilon} + \mathcal{O}\left(\varepsilon^5\right), \tag{18}$$

where the 4 comes from the trace of the unit matrix in the defining representation of $SO(4)$. Both expressions (5) and (18) reflect the fact that when we zoom in on the neighborhood of some point $x \in M$, once the resolution scale ε falls below $\varepsilon \lesssim 1/\sqrt{|R(x)|}$, where $|R(x)|$ denotes the magnitude of the largest curvature at x , the neighborhood will look ever more flat. However, such a flattening is not expected when we zoom in on a quantum geometry, obtained from the continuum limit of the gravitational path integral (6), because of its nowhere differentiable character. As a consequence, a general holonomy $U(\gamma)$ will not be of the form "unit matrix plus a small perturbation," and an analogous statement holds for the corresponding Wilson loop.

其中数字 4 来自 $SO(4)$ 基础表示中单位矩阵的迹。式 (5) 和 (18) 都反映了一个事实: 当我们放大某点 $x \in M$ 的邻域，一旦分辨率尺度 ε 低于 $\varepsilon \lesssim 1/\sqrt{|R(x)|}$ (其中 $|R(x)|$ 是 x 处最大曲率的大小)，该邻域会越来越接近平直。但对于引力路径积分 (6) 连续极限得到的量子几何，由于其处处不可微的性质，放大时并不会出现这种平直化。因此，一般的和乐 $U(\gamma)$ 不会呈现“单位矩阵加小扰动”的形式，对应的威尔逊环也同理。

This leads one to consider non-infinitesimal Wilson loops, exploring the possibility that the inherent integration along a finite loop γ may implement some effective "averaging out" of the short-distance quantum behavior of the underlying quantum geometry. Here, one encounters another difficulty, related to the non-abelian character of the holonomy of the Levi-Civita connection. In a classical context, if the holonomies of closed curves take values in an abelian group, one can by a suitable version of Stokes' theorem relate the path-ordered exponential of the connection to the exponential of a surface integral of its curvature. However, holonomies on a general curved manifold take values in all of $SO(4)$, in which case there is no obvious way to relate finite Wilson loops to an integrated or coarse-grained form of local curvature. (There are known exceptions when $\left(M, g_{\mu\upsilon}\right)$ has special isometries and the loops lie inside totally geodesic surfaces [21].) Imitating the abelian construction leads to a gravitational version of what is sometimes called the non-abelian Stokes' theorem, which in Riemannian geometry was already considered almost 100 years ago [22]. It is not particularly useful for our purposes since Riemann curvature occurs only in a highly nonlocal form inside the area integral appearing in the theorem, due to the presence of surface ordering, a two-dimensional analogue of path ordering, which is needed because of the noncommutative nature of the connection and its associated curvature (see, e.g., [21] and references therein).

这促使我们考虑非无穷小威尔逊环，探究有限回路 γ 上的固有积分可能会对底层量子几何的短距离量子行为起到有效的“平均”作用。在此我们遇到另一个困难，它与列维-奇维塔联络和乐的非阿贝尔性质有关。在经典语境下，如果闭合曲线的和乐取值在阿贝尔群中，我们可以通过适当版本的斯托克斯定理，把联络的路径序指数和联络曲率的面积分指数联系起来。然而，一般弯曲流形上的和乐可以取遍整个 $SO(4)$ 中的值，此时不存在显而易见的方法将有限威尔逊环和局域曲率的积分或粗粒化形式联系起来。(当 $(M, g_{\mu\nu})$ 具有特殊等距性，且回路位于全测地曲面内时，存在已知例外 [21]。) 仿照阿贝尔情形的构造，得到了有时被称为非阿贝尔斯托克斯定理的引力版本，这在近 100 年前就已经在黎曼几何中被研究过 [22]。它对我们的研究目的并没有太大用处，因为由于曲面有序 (路径有序在二维的类比，它因联络及其曲率的非对易性质而必须存在)，黎曼曲率仅以高度非局域的形式出现在定理的面积分中。(参见例如 [21] 及其中参考文献)

Despite these obstacles, we cannot rule out that in nonperturbative quantum gravity, defined by CDT or otherwise, suitably chosen Wilson loops could display a semi-classical behavior matching an expression like (5), allowing us to extract an effective curvature at some scale. The other possibility is that observables based on Wilson loops are not useful as measures of (renormalized) curvature in this context.

尽管存在这些障碍，我们不能排除: 在由 CDT 或其他方法定义的非微扰量子引力中，适当选取的威尔逊环可以展现出与 (5) 式这类表达式匹配的半经典行为，从而允许我们提取特定尺度下的有效曲率。另一种可能是，基于威尔逊环的可观测量并不适合作为该背景下 (重整化) 曲率的度量。

Quantum Ricci Curvature

量子里奇曲率

By contrast, the quantum Ricci curvature, introduced in [4], has already been used to construct well-behaved curvature observables in nonperturbative quantum gravity. It is based on a quasi-local rods-and-clocks construction using geodesic distances and volumes and so far is applicable on metric spaces with positive definite signature, in particular, the Wick-rotated geometries of the CDT formulation. As explained in greater detail below, it involves a comparison of geodesic spheres of radius δ and thereby yields a generalized notion of Ricci curvature associated with a coarse-graining scale δ . This is convenient from a physics point of view, where one is often interested in the behavior of physical observables as a function of a chosen length scale. Although the QRC can also be implemented on classical Riemannian spaces, where it reproduces the standard Ricci curvature in the limit $\delta \to 0$, a remarkable feature of the quantum Ricci curvature is that it does not require a smooth structure, tensor calculus or parallel transport, and can therefore be applied on a wide range of beyond-classical geometries.

相比之下，文献 [4] 中引入的量子里奇曲率已被用于在非微扰量子引力中构造性质良好的曲率可观测量。它基于利用测地线距离和体积构造的准局部“杆钟”方案，目前适用于具有正定符号的度量空间，尤其是 CDT 表述的威克旋转几何。正如下文更详细的说明，该方法通过对比半径为 δ 的测地球，得到了与粗粒化尺度 δ 关联的广义里奇曲率概念。从物理学角度来看这十分便利，因为物理学中通常很关注物理可观测量随所选长度尺度变化的行为。尽管 QRC 也可以在经典黎曼空间上实现，并且在 $\delta \to 0$ 极限下能够重现标准里奇曲率，但量子里奇曲率的一个突出特点是它不需要光滑结构、张量微积分或平行传输，因此可以广泛应用于各类超越经典的几何结构。

Construction and Implementation

构造与实现

Given a D -dimensional metric measure space, the main idea underlying the QRC is to compare the distance $\bar{d}$ of two nearby $(D-1)$ -dimensional spheres S_p and $S_{p'}$ of radius ε (as point sets) with the distance δ of their centers p and p' (Fig. 7). The key geometric insight is then expressed by the sphere-distance criterion: ”On a space with positive (negative) Ricci curvature, the sphere distance $\bar{d}$ is smaller (bigger) than the centre distance δ .” This observation has been used in mathematics to define a generalized notion of curvature, now known as Ollivier-Ricci curvature [23] (see also [24]), which has since become a much-used tool in discrete mathematics and graph theory [25, 26]. It has also been used in graph-theoretic models for quantum gravity (see [27, 28] and references therein).

给定一个 D 维度量测度空间，量子 Ricci 曲率的核心思想是: 比较两个半径为 ε 的相邻 $(D-1)$ 维球面 (作为点集) S_p 和 $S_{p'}$ 之间的距离 $\bar{d}$ ，与它们球心 p 和 p' 之间的距离 δ (图 7)。核心几何洞见由球面距离准则表述: “在正 (负)Ricci 曲率空间中，球面距离 $\bar{d}$ 小于 (大于) 球心距离 δ 。” 这一观测已在数学中用于定义曲率的广义概念，即如今所知的 Ollivier-Ricci 曲率 [23](另见 [24])，此后它成为离散数学和图论中被广泛使用的工具 [25, 26]，也已应用于量子引力的图论模型中 (见 [27, 28] 及其中参考文献)。

The quantum Ricci curvature [4, 5] is inspired by this work and has adapted it in a nontrivial way to the ensemble of piecewise flat geometries that forms the configuration space of the nonperturbative gravitational path integral (7) or, more precisely, its analytical continuation (10). Firstly, in order to have only a single length scale involved, one sets $\varepsilon=\delta$, such that the two spheres overlap partially. Secondly, one uses as the distance $\bar{d}$ the average sphere distance (instead of the L^1 -transportation (or Wasserstein) distance of [23], which is very expensive computationally), which in standard Riemannian continuum language is given by

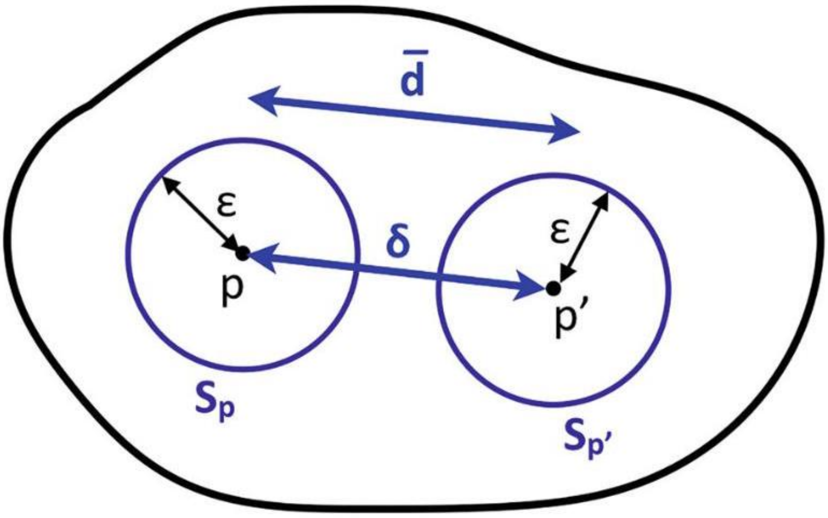
量子 Ricci 曲率 [4, 5] 受这项工作启发，以非平凡方式将其适配到分段平坦几何系综，该系综构成了非微扰引力路径积分 (7)，更准确地说，是其解析延拓 (10) 的构型空间。首先，为了仅涉及单个长度标度，我们设定 $\varepsilon=\delta$ ，使得两个球面部分重叠。其次，我们采用平均球面距离作为距离 $\bar{d}$ (而非文献 [23] 中 L^1 运输距离，即 Wasserstein 距离，该距离计算成本极高)，用标准黎曼连续统语言表述为:

$$\bar{d}\left(S_p, S_{p'}\right) := \frac{1}{\operatorname{vol}\left(S_p\right)} \frac{1}{\operatorname{vol}\left(S_{p'}\right)} \int_{S_p} d^{D-1} q \sqrt{\det h} \int_{S_{p'}} d^{D-1} q' \sqrt{\det h'}\, d_g\left(q, q'\right),$$

(19)

Fig. 7 A generalized notion of Ricci curvature is obtained by comparing the distance $\bar{d}$ of two ε -spheres S_p and $S_{p'}$ with the distance δ of their centers p and p'

图 7 通过比较两个 ε 球面 S_p 和 $S_{p'}$ 的距离 $\bar{d}$ 与它们球心 p 和 p' 的距离 δ ，得到 Ricci 曲率的广义概念



where $d_g(q, q')$ denotes the geodesic distance of the points q and q' and h and h' are the metrics induced on the two spheres. The quasi-local quantum Ricci curvature K_q at scale δ is then defined in terms of the quotient of the average sphere distance (19) and the center distance as

其中 $d_g(q, q')$ 表示点 q 和 q' 的测地距离，h 和 h' 是两个球面上诱导的度量。标度 δ 处的拟局部量子 Ricci 曲率 K_q 由平均球面距离 (19) 与球心距离的商定义如下:

$$\bar{d}\left(S_p, S_{p'}\right)/\delta = c_q\left(1 - K_q\left(p, p'\right)\right), \ \delta = d_g\left(p, p'\right), \tag{20}$$

where c_q is a non-universal δ -independent constant with $0 < c_q < 3$, depending on the type and dimension of the metric space, and K_q captures the non-constant remainder. Note that K_q is dimensionless by construction. It is therefore a curvature in units of some inverse length scale, which can be identified with $1/\delta$, at least for sufficiently small K_q . When evaluating the quotient (20) on a smooth Riemannian space for infinitesimal δ , one finds that K_q to lowest, quadratic order in δ contains $\mathrm{Ric}(v, v) = R_{ij}v^i v^j$, the usual Ricci tensor contracted with the unit vector v between p and p' . A computation in Riemann normal coordinates yields [5]

其中 c_q 是不依赖于 δ 的非普适常数，满足 $0 < c_q < 3$ ，取决于度量空间的类型和维数，K_q 包含了非恒定余项。注意根据构造，K_q 是无量纲的，因此它是某逆长度标度单位下的曲率，至少对于足够小的 K_q ，该逆标度可等同于 $1/\delta$ 。在光滑黎曼空间中对无穷小 δ 计算商式 (20) 可发现，K_q 在 δ 的最低二次阶包含 $\mathrm{Ric}(v, v) = R_{ij}v^i v^j$ ，即常规 Ricci 张量与 p 和 p' 之间单位向量 v 的缩并。黎曼法坐标下的计算给出 [5]:

$$\frac{\bar{d}}{\delta} = \begin{cases} 1.5746 + \delta^2\left(-0.1440 Ric(v, v)\right) + \mathcal{O}\left(\delta^3\right), & D = 2, \\ 1.6250 + \delta^2\left(-0.0612 Ric(v, v) - 0.0122 R\right) + \mathcal{O}\left(\delta^3\right), & D = 3, \\ 1.6524 + \delta^2\left(-0.0469 Ric(v, v) - 0.0067 R\right) + \mathcal{O}\left(\delta^3\right), & D = 4. \end{cases} \tag{21}$$

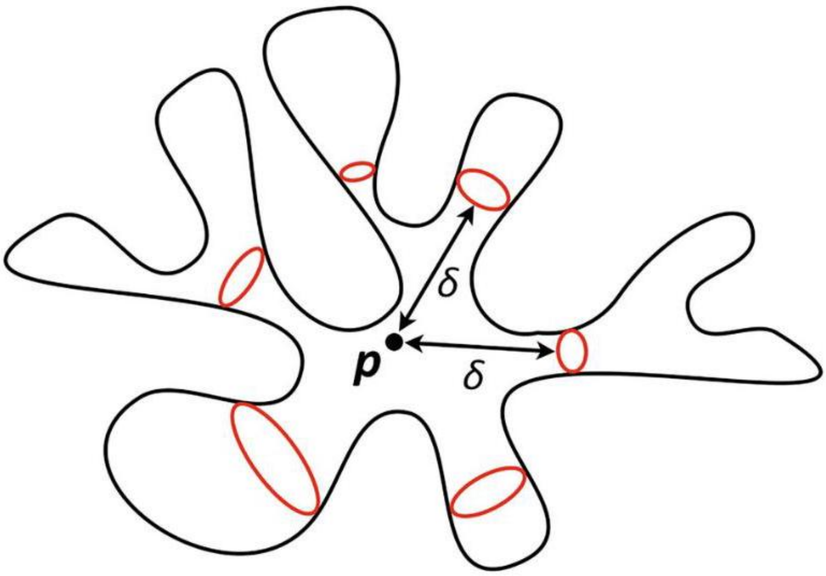
Of course, the main motivation for this construction is the nonperturbative quantum theory, where the quantum Ricci curvature provides us with a notion of a Ricci curvature at coarse-graining scale δ , for non-infinitesimal δ . The standard notion of (discrete, geodesic) distance used on (C)DT configurations is the link distance $d(q, q')$, defined as the length of the shortest path along lattice links between two given vertices q and q' . Alternatively, one may use the dual link distance, defined as the length of the shortest path along dual lattice links between two given dual vertices. In the continuum limit, these two choices should lead to equivalent results, but on finite lattices, one can be more convenient than the other (see section "CDT Quantum Gravity in D = 4" for an example). A straightforward implementation of the average sphere distance (19) on a triangulation is

当然，该构造的主要动机来自非微扰量子理论，在非微扰量子理论中，量子里奇曲率为我们提供了粗粒化尺度 δ 下的里奇曲率概念，且 δ 不必是无穷小。(因果动态三角剖分) 构型中使用的标准 (离散测地线) 距离是链距离 $d(q, q')$ ，其定义为两个给定顶点 q 与 q' 之间沿晶格链的最短路径长度。也可选用对偶链距离，其定义为两个给定对偶顶点之间沿对偶晶格链的最短路径长度。在连续极限下，这两种选择应当给出等价结果，但在有限晶格上，其中一种会比另一种更方便 (例子参见“D=4 下的 CDT 量子引力”一节)。平均球面距离 (19) 在三角剖分上的直接实现为

$$\bar{d}\left(S_p^\delta, S_{p'}^\delta\right) = \frac{1}{N_0\left(S_p^\delta\right)} \frac{1}{N_0\left(S_{p'}^\delta\right)} \sum_{q \in S_p^\delta} \sum_{q' \in S_{p'}^\delta} d(q, q'), \; d(p, p') = \delta, \tag{22}$$

Fig. 8 On this two-dimensional smooth manifold with S^2 -topology, the one-dimensional "sphere" S_p^δ with center point p is not a single one-sphere, but consists of six disjoint components

图 8 在这个具有 S^2 拓扑的二维光滑流形上，以点 p 为中心的一维“球面” S_p^δ 不是单个一维球面，而是由六个不相交的分支组成



where the sphere S_p^δ of (integer) radius δ around a given lattice vertex p is defined as the set of all vertices at link distance δ from p . Note that for $\delta > 1$, this set in general does not form a topological $(D-1)$ -sphere, in the sense that the vertex set S_p^δ does not span a triangulated sphere, but instead consists of several components, as illustrated by the two-dimensional example of Fig. 8. On highly nonclassical quantum geometries, this kind of branching behavior can occur even on the smallest scales.

其中，给定晶格顶点 p 周围 (整数) 半径 δ 的球面 S_p^δ 被定义为所有与 p 相距链距离 δ 的顶点集合。注意，一般而言，对于 $\delta > 1$ ，该集合并不构成拓扑 $(D-1)$ 球面——顶点集合 S_p^δ 不能张成一个三角剖分球面，而是由多个分支组成，如图 8 的二维示例所示。在高度非经典量子几何中，这种分支行为即使在最小尺度上也会发生。

In situations where one is not interested in directional information about the curvature, one can use a variant of the QRC where one lets the two spheres of Fig. 7 coincide, i.e., sets $\delta = 0$ instead of $\delta = \varepsilon$ to obtain a ”rotationally symmetric” set-up. The analogue of the continuum expansion (21) in this case is structurally similar, but with different constants and only dependent on the Ricci scalar and (at higher orders) its covariant derivatives. In the construction of the curvature profile in the next subsection, we will stick to the standard definition of the QRC and average over directions according to need.

在不需要曲率方向信息的场景中, 可以使用量子里奇曲率的一个变体: 令图 7 中的两个球面重合, 即设 $\delta = 0$ 而非 $\delta = \varepsilon$ ，得到一个“旋转对称”的设置。这种情况下，连续展开 (21) 的对应式在结构上相似，但常数不同，且仅依赖于里奇标量 (高阶项中依赖于里奇标量的协变导数)。在下一小节构造曲率轮廓时，我们将沿用量子里奇曲率的标准定义，根据需要对方向做平均。

The quantum Ricci curvature has been explored and tested on a variety of classical spaces in two and three dimensions [4]. On constantly curved Riemannian spaces, there are exact integral expressions for the average sphere distance (19). They can be evaluated numerically and serve as benchmarks for the interpretation of curvature profiles in nonperturbative applications, as we will see below. Implementing the QRC on tessellations of flat spaces and a class of equilateral triangulations obtained from Delaunay triangulations, which represent random approximations of constantly curved continuum spaces, gives important insights into the nature of lattice artifacts. Characteristic for all lattices is the presence of a region $\delta \lesssim 5$ in link units, where the normalized average sphere distance $\bar{d}/\delta$ of (20) is dominated by unphysical short-distance lattice artifacts, with an initial ”overshoot.” Furthermore, as already remarked earlier, the δ -independent term c_q in $\bar{d}/\delta$ is not universal, but depends on the type of lattice. (For a given vertex, c_q can also depend on the lattice direction, as was noted for the regular flat lattices of [4]. The reason is the highly anisotropic behavior of the link distance on such lattices; for example, a ”sphere” S_p^δ based at a vertex p of a 2D regular square lattice is square-shaped rather than round.) Both of these properties are illustrated in Fig. 9, which shows the behavior of the quotient $\bar{d}/\delta$ as a function of the link distance δ on two different lattice discretizations of two-dimensional flat space. Within measuring accuracy, both are compatible with a vanishing QRC, $K_q = 0$. These examples also illustrate that the constant c_q of relation (20), which in the continuum is given by $c_q = \lim_{\delta \to 0} \bar{d}/\delta$, has to be defined appropriately on piecewise flat spaces, at some reference point δ outside the region of lattice artifacts, typically chosen as $\delta = 5$ or $\delta = 6$.

量子里奇曲率已在多种二维和三维经典空间中得到探索与检验 [4]。在常曲率黎曼空间中，平均球面距离存在精确积分表达式 (19)。它们可通过数值计算得到，并作为非微扰应用中曲率轮廓解读的基准，我们下文会对此进行说明。在平坦空间的 Tessellation 剖分以及由 Delaunay 三角剖分得到的一类等边三角化 (这类三角化是常曲率连续空间的随机近似) 上实现量子里奇曲率，能为我们提供关于晶格赝象本质的重要洞见。所有晶格的共同特征是，以链接为单位存在一个区域 $\delta \lesssim 5$，其中式 (20) 的归一化平均球面距离 $\bar{d}/\delta$ 由非物理的短距离晶格赝象主导，初始会出现“超调”。此外，正如前文所述，$\bar{d}/\delta$ 中独立于 δ 的项 c_q 并非普适的，它依赖于晶格的类型。(对给定顶点，c_q 还可依赖于晶格方向，正如文献 [4] 中的规则平坦晶格所指出的。原因是这类晶格上链接距离具有高度各向异性；例如，二维规则正方形晶格中，以顶点 p 为心的“球面” S_p^δ 是方形而非圆形。) 上述两种性质都在图 9 中得到了展示，图 9 展示了二维平坦空间两种不同晶格离散化下，商 $\bar{d}/\delta$ 随链接距离 δ 的变化行为。在测量精度范围内，二者都与量子里奇曲率为零 $K_q = 0$ 相容。这些例子还说明，关系式 (20) 中的常数 c_q (连续情形下由 $c_q = \lim_{\delta \to 0} \bar{d}/\delta$ 给出) 必须在分段平坦空间上、在晶格赝象区域外的某个参考点 δ 处适当定义，通常选 $\delta = 5$ 或 $\delta = 6$。

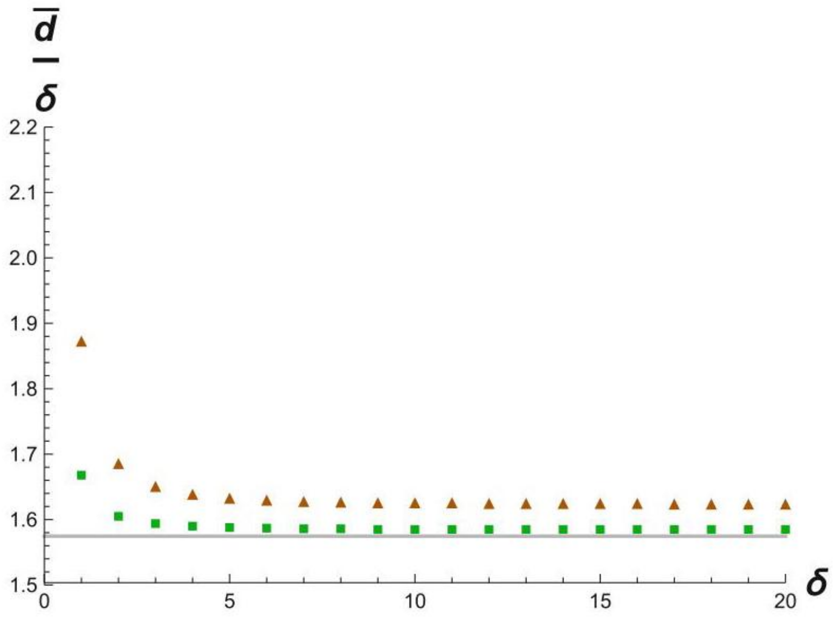


Fig. 9 Normalized average sphere distance $\bar{d}\left(S_p^\delta, S_{p'}^\delta\right)/\delta$ on regular 2D flat lattices, averaged over relative orientations of p and p'. Triangles and squares mark the data points for the square and hexagonal lattices, respectively. The straight horizontal line of flat 2D continuum space is included for comparison [4]

图 9 二维规则平坦晶格上的归一化平均球面距离 $\bar{d}\left(S_p^\delta, S_{p'}^\delta\right)/\delta$，结果已对 p 和 p' 的相对取向取平均。三角形和正方形分别标记正方形晶格和六角晶格的数据点。图中加入了二维平坦连续空间的水平直线作为对比 [4]

Curvature Observables: The Curvature Profile

曲率可观测量: 曲率轮廓

Neither the average sphere distance (19), (22), nor the quantum Ricci curvature extracted from it are observables, since they still depend on a specific point pair (p, p') . The most straightforward way of constructing a diffeomorphism-invariant observable is by averaging the average sphere distance $\bar{d}$ over all positions p and p' of the two circle centers while keeping their distance δ fixed. Adopting again a continuum language, the average on a given compact Riemannian manifold $\left(M, g_{\mu\nu}\right)$ of the average sphere distance at the scale δ is given by

无论是平均球面距离 (19)、(22)，还是从中提取出的量子里奇曲率都不是可观测量，因为它们仍然依赖于特定的点对 (p, p') 。构造微分同胚不变可观测量最直接的方法，是固定两圆心距离 δ 不变，对所有圆心位置 p 和 p' 的平均球面距离 $\bar{d}$ 求平均。再次采用连续语言，给定紧致黎曼流形 $\left(M, g_{\mu\nu}\right)$ 上尺度 δ 处平均球面距离的平均值为

$$\bar{d}_{\mathrm{av}}(\delta) := \frac{1}{Z_\delta}\int_M d^D p\sqrt{\det g}\int_M d^D p'\sqrt{\det g}\bar{d}\left(S_p^\delta, S_{p'}^\delta\right)\delta^{Dir}\left(d_g(p,p'),\delta\right),$$

(23)

where δ^{Dir} denotes the Dirac delta function and the normalization factor Z_δ is given by

其中 δ^{Dir} 表示狄拉克 δ 函数，归一化因子 Z_δ 由下式给出

$$Z_\delta = \int_M d^D p\sqrt{\det g}\int_M d^D p'\sqrt{\det g}\delta^{Dir}\left(d_g(p,p'),\delta\right). \tag{24}$$

Since the integration in Eq. (23) includes an averaging over directions, it only allows us to extract an (averaged) quantum Ricci scalar $K_{\mathrm{av}}(\delta)$. This scalar quantity appears in the δ -dependent curvature profile [29], which is defined as the quotient

由于式 (23) 中的积分包含方向平均，我们只能从中提取一个 (平均) 量子里奇标量 $K_{\mathrm{av}}(\delta)$ 。该标量出现在依赖 δ 的曲率轮廓 [29] 中，曲率轮廓定义为如下商

$$\bar{d}_{\mathrm{av}}(\delta)/\delta =: c_{\mathrm{av}}\left(1 - K_{\mathrm{av}}(\delta)\right), \tag{25}$$

where the constant c_{av} is given by $c_{\mathrm{av}} = \lim_{\delta\to 0}\bar{d}_{\mathrm{av}}/\delta$. The curvature profile is a nonlocal curvature observable characterizing a given curved manifold.

其中常数 c_{av} 由 $c_{\mathrm{av}} = \lim_{\delta\to 0}\bar{d}_{\mathrm{av}}/\delta$ 给出。曲率轮廓是描述给定弯曲流形的非局部曲率可观测量。

The curvature profile is not an observable that one would naturally consider in a classical context, where one is usually interested in resolving local curvature properties of selected solutions to the Einstein equations. The situation is different in the quantum theory, where one is primarily interested in the quantum geometry of the unique nonperturbative vacuum. One may expect this quantum geometry to have some approximate global symmetries, at least on suitably coarse-grained scales, although this conjecture still needs to be verified explicitly (see also the discussion of the curvature profile of the de Sitter-like vacuum state of 4D CDT quantum gravity in section ”CDT Quantum Gravity in D = 4” and [14]). If such symmetries are indeed present, the difference between local curvature properties and their space(time) averages may not be all that big.

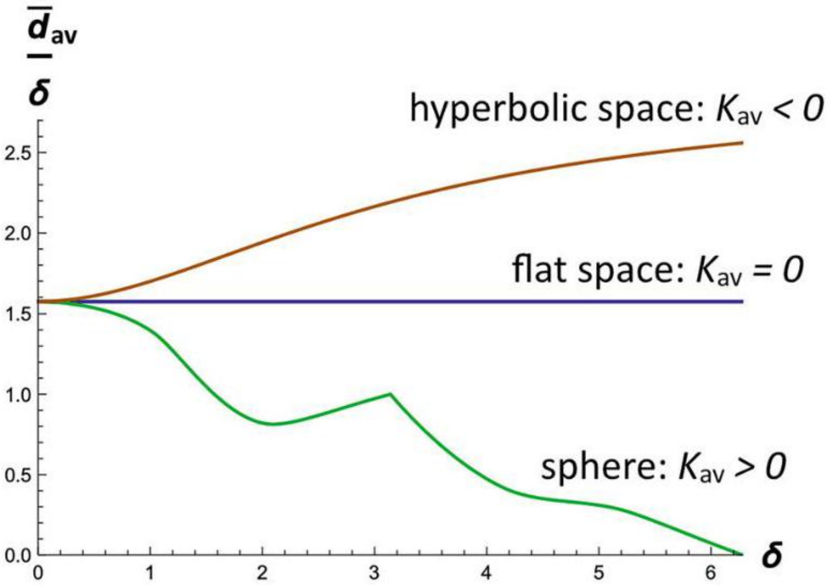
曲率轮廓并非经典语境中会自然被研究的可观测量，经典领域通常关注求解爱因斯坦方程得到的特定解的局部曲率性质。量子理论中的情况则不同，我们主要关注唯一非微扰真空的量子几何。可以预期该量子几何存在某种近似整体对称性，至少在合适粗粒化的尺度上是如此，不过这一猜想仍需明确验证 (另见“D=4 维的 CDT 量子引力”章节以及文献 [14] 对 4 维 CDT 量子引力类德西特真空态曲率轮廓的讨论)。如果这类对称性确实存在，那么局部曲率性质与其时空平均值之间的差异或许并没有那么大。

In order to interpret (the expectation values of) curvature profiles obtained in the quantum theory, it will be helpful to compare them to specific classical curvature profiles. In line with the argument of the previous paragraph, a natural reference point are the constantly curved differentiable manifolds of positive, negative, or vanishing curvature, that is, spherical, hyperbolic, or flat spaces in any dimension D . Note that in these cases, no averaging over M is needed to obtain the classical curvature profile, since the average sphere distance (19) depends only on the distance δ and not on the location of p and p' . The curvature profiles of spaces of constant curvature were computed in [4] and are shown in Fig. 10 for $D=2$; the analogous curvature profiles in higher dimensions look qualitatively similar. Note that the characteristic overall shape of the curvature profile for S^2 has to do with its global properties and perfect roundness. (For example, the kink at $\delta=\pi$ corresponds to the situation where the two sphere centers p and p' are antipodes and their associated ”circles” S_p^δ and $S_{p'}^\delta$ of radius δ are degenerate and themselves correspond to points. On a space that only approximates a sphere in some loose, average sense, the global features of the curvature profile may look very different.) In practice, when comparing with quantum measurements, only the initial regions of these curves are used, where δ is much smaller than the linear extension of the (quantum) space under consideration. The main feature of $\bar{d}_{\mathrm{av}}/\delta$ one looks for in this range is whether it increases, decreases, or stays constant in δ , corresponding to a negative, positive, or vanishing average quantum Ricci scalar K_{av} .

为了解释量子理论中得到的曲率轮廓 (的期望值)，将其与特定经典曲率轮廓对比会很有帮助。根据上一段的论述，自然的参考对象就是具有正曲率、负曲率或零曲率的常曲率微分流形，也就是任意维 D 下的球面空间、双曲空间或平坦空间。注意这类情况下，得到经典曲率轮廓不需要对 M 平均，因为平均球面距离 (19) 仅依赖距离 δ ，与 p 和 p' 的位置无关。常曲率空间的曲率轮廓已在文献 [4] 中计算完成，图 10 展示了 $D=2$ 对应的结果；更高维下的同类曲率轮廓定性来看是相似的。注意，S^2 曲率轮廓的特征整体形状与其全局性质和完美圆性有关。(例如，$\delta=\pi$ 处的扭结对应两个球心 p 和 p' 互为对跖点的情况，此时它们对应的半径为 δ 的“球面”S_p^δ 和 $S_{p'}^\delta$ 退化，退化为点。在仅于松散平均意义上近似球面的空间中，曲率轮廓的全局特征可能会大不相同。) 实际应用中，与量子测量对比时仅会用到这些曲线的初始区域，此时 δ 远小于所研究 (量子) 空间的线性尺度。在该区间内，我们关注 $\bar{d}_{\mathrm{av}}/\delta$ 的核心特征是它随 δ 增大是增加、减少还是保持不变，分别对应负、正或零的平均量子里奇标量 K_{av} 。

Fig. 10 Curvature profiles $\bar{d}_{\mathrm{av}}(\delta)/\delta$ of constantly curved Riemannian spaces in $D=2$ [4]. The curvature radius has been set to 1

图 10 常曲率黎曼空间的曲率轮廓 $\bar{d}_{\mathrm{av}}(\delta)/\delta$ ，出自 $D=2$ [4]。曲率半径已设为 1



The classical curvature profile (25) is an observable, and its expectation value $\langle \bar{d}_{av}(\delta)/\delta \rangle$ can be determined numerically in (C)DT or other formulations of quantum gravity. Note that in a lattice approach like CDT, all distances are given in dimensionless lattice units, which can be converted into dimensionful units by invoking the lattice spacing a , introduced in section "Deficit Angle Curvature in Dynamical Triangulations". Recall that the dimensionless $K_{av}(\delta)$ has the interpretation of a curvature in units of $1/\delta$, both classically and in the non-perturbative quantum theory. Assuming for the moment that δ scales canonically, that is, proportional to a or, equivalently, proportional to $N_D^{1/D}$, one can extract a dimensionful, renormalized quantum Ricci scalar $K^r(\delta_{ph})$, which depends on a physical, renormalized length scale $\delta_{ph} := a\delta$, via

经典曲率轮廓 (25) 是一个可观测量，其期望值 $\langle \bar{d}_{av}(\delta)/\delta \rangle$ 可以在因果动态三角剖分 (CDT) 或其他量子引力表述中通过数值确定。注意，在 CDT 这类格点方法中，所有距离都以无量纲格点单位给出，可通过引入“动态三角剖分中的亏角曲率”一节介绍的格点间距 a 转换为量纲单位。需要说明的是，无论在经典还是非微扰量子理论中，无量纲量 $K_{av}(\delta)$ 都被解释为以 $1/\delta$ 单位下的曲率。暂时假设 δ 满足正则标度，即正比于 a ，等价地，正比于 $N_D^{1/D}$ ，我们就可以通过下式提取有量纲的重整化量子里奇标量 $K^r(\delta_{ph})$ ，它依赖于物理的重整化长度标度 $\delta_{ph} := a\delta$

$$K_{av}(\delta) =: \delta^2 a^2 K^r(\delta_{ph}) = (\delta_{ph})^2 K^r(\delta_{ph}), \tag{26}$$

in the limit as $a \to 0$. In case δ scales non-canonically, the renormalized curvature will scale accordingly and also non-canonically.

取 $a \to 0$ 下的极限。若 δ 不满足正则标度，重整化曲率也会相应地发生非正则标度。

Averaging Properties of the QRC

QRC 的平均性质

Moving away from constantly curved spaces, another interesting question is to what extent the classical curvature profile is sensitive to an inhomogeneous distribution of curvature. From the viewpoint of the quantum theory, it is desirable that any notion of curvature has good averaging properties, unlike the deficit angle curvature, which is a local curvature defined at the cutoff scale and already singular at the classical level, in the sense of being concentrated on (D-2)-dimensional hinges, as described in section ”Curvature as Deficit Angles”. As we saw in section ”Deficit Angle Curvature in Dynamical Triangulations”, in a continuum limit, these curvature singularities just become denser and denser, without ”averaging out.”

脱离常曲率空间后，另一个值得研究的问题是经典曲率轮廓对曲率的非均匀分布敏感程度如何。从量子理论的角度来看，任何曲率概念都理想地具备良好的平均性质，这一点和亏角曲率不同: 亏角曲率是定义在截断尺度上的局部曲率，它集中在 (D-2) 维铰链上，在经典层面就已经是奇异的，这一点我们在“作为亏角的曲率”一节中已有描述。正如我们在“动态三角剖分中的亏角曲率”一节中看到的，在连续极限下，这些曲率奇点只会变得越来越密集，不会被“平均掉”。

One class of triangulations whose curvature profiles have been studied involves 2D Delaunay triangulations approximating constantly curved spaces. Delaunay triangulations are random, piecewise flat spaces, which by construction closely approximate their smooth counterparts in terms of their curvature properties. One proceeds by constructing via a Poisson disc sampling a point set P in the constantly curved model space in question, with a chosen minimal distance $d_{\min}$ between any two of its points. One then constructs the Delaunay triangulation that has P as its vertices and subsequently sets all edge lengths to 1 to obtain an equilateral triangulation on which the QRC can be implemented in a straightforward way (see [4] for details). Up to an overshoot for small $\delta \lesssim 5$ and a vertical shift due to the non-universal constant c_q , features which were already observed in lattice representations of flat space (cf. Fig.9), the volume profiles for these random triangulations closely resemble those of their constantly curved continuum counterparts.

有一类曲率轮廓已被研究的三角剖分, 是用于近似常曲率空间的二维 Delaunay 三角剖分。Delaunay 三角剖分属于分片平坦的随机空间，根据构造方法，其曲率性质可以很好地逼近对应的光滑空间。具体步骤是，通过泊松圆盘采样在目标常曲率模型空间中生成点集 P ，任意两点间满足选定的最小距离 $d_{\min}$ 。随后构造以 P 为顶点的 Delaunay 三角剖分，再将所有边长设为 1 得到等边三角剖分，即可直接在该剖分上计算量子 Ricci 曲率 (细节见文献 [4])。除了小 $\delta \lesssim 5$ 会出现超调，且非普适常数 c_q 会带来一个纵向偏移——这两个特征在平坦空间的晶格表示中就已被观测到 (参见图 9)，这些随机三角剖分的体积轮廓与常曲率连续空间对应轮廓非常接近。

Another type of classical geometry whose curvature profile has been studied is a two-dimensional space of spherical topology with isolated conical singularities, more specifically, a regular polygon that forms the surface of one of the Platonic solids [29]. The polygon with the largest conical defects is the surface of a tetrahedron, which is flat except at its four corners, with a deficit angle of π each. In terms of curvature (in)homogeneity, it is the extreme opposite of a round sphere, on which the same total Gaussian curvature (4π by the Gauss-Bonnet theorem) is distributed completely homogeneously. In addition, a lot is known about geodesics on the tetrahedron, which allows one to construct the geodesic circles that are key to the computation of the QRC [29].

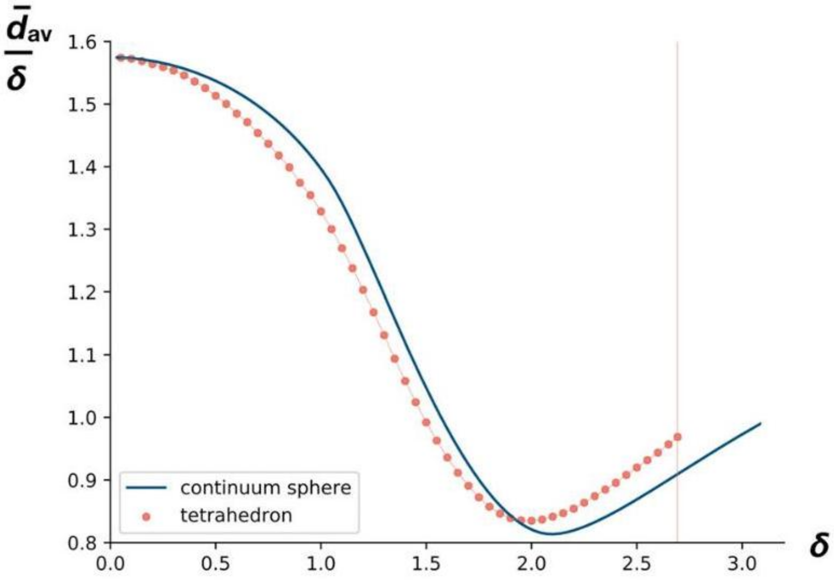
另一类已研究过曲率分布的经典几何是带有孤立锥形奇点的二维球面拓扑空间，具体来说，是构成柏拉图多面体其中一个面的正多边形 [29]。锥形缺陷最大的多边形是正四面体表面，它除四个角外整体平坦，每个角的亏缺角均为 π 。就曲率 (非) 均匀性而言，它与圆球是完全相反的极端情况: 根据高斯-博内定理，圆球的总高斯曲率与正四面体相同，为 4π ，但该总曲率在圆球上完全均匀分布。此外，人们对正四面体上的测地线已有充分研究，因此可以构造出计算 QRC 的关键要素——测地圆 [29]。

The curvature profiles for the tetrahedral surface and for a sphere of the same area are shown in Fig. 11. The reference curve for the sphere coincides with that of Fig. 10 for $\delta \leq \pi$. The two curvature profiles clearly differ, but not by very much, testifying to the averaging property of the quantum Ricci curvature. This is a reassuring feature, but the real test of the QRC is its application in the nonperturbative quantum theory, which will be the subject of section ”Quantum Ricci Curvature: Quantum Applications”. At the same time, these results emphasize the global nature of the curvature profile $\bar{d}_{\mathrm{av}}(\delta)/\delta$. Similar to the volume profile $V_3(\tau)$, which denotes (the expectation value of) the spatial volume as a function of proper time in 4D CDT quantum gravity [2, 3, 30] , also the curvature profile needs to be complemented by other observables, like correlators or homogeneity measures [31] to obtain a finer-grained understanding of the underlying quantum geometry.

四面体曲面和相同面积球体的曲率轮廓如图 11 所示。球体的参考曲线与图 10 中 $\delta \leq \pi$ 的参考曲线重合。两条曲率轮廓存在明显差异，但差异并不大，这证明了量子里奇曲率的平均性质。这是一个令人安心的特征，但量子里奇曲率 (QRC) 真正的检验在于它在非微扰量子理论中的应用，这将是“量子里奇曲率: 量子应用”一节的主题。同时，这些结果凸显了曲率轮廓 $\bar{d}_{\mathrm{av}}(\delta)/\delta$ 的全局性质。类似于体积轮廓 $V_3(\tau)$ (它表示四维因果动态三角剖分 (CDT) 量子引力 [2, 3, 30] 中，空间体积的期望值是固有时的函数)，曲率轮廓也需要其他可观测量 (比如关联函数或均匀性度量 [31]) 作为补充，才能更细致地理解其底层量子几何。

Fig. 11 Measured curvature profile $\bar{d}_{\mathrm{av}}(\delta)/\delta$ of the surface of a tetrahedron, compared to that of a two-sphere of the same area. The vertical line at $\delta = 2.694$ marks the edge length of the tetrahedron [29]

图 11 四面体表面的测量曲率轮廓 $\bar{d}_{\mathrm{av}}(\delta)/\delta$ ，与相同面积二维球面的测量曲率轮廓对比。位于 $\delta = 2.694$ 的竖线标记了四面体的边长 [29]



Quantum Ricci Curvature: Quantum Applications

量子里奇曲率: 量子应用

The quantum Ricci curvature, which was introduced in the previous section, gives us a new tool to characterize and quantify quantum geometry, including in regimes far away from classicality. It is not known a priori what kind of nontrivial quantum behavior the QRC can exhibit, but there is no reason in principle why it should be any less complex than that of its classical counterpart. In the context of nonperturbative models based on dynamical triangulations, a natural first testing ground for the QRC are toy models in two dimensions. Pure gravity models in 2D, whose action consists of a (topological) Einstein-Hilbert term and a cosmological-constant term, fall into two distinct universality classes, depending on their metric signature. CDT quantum gravity lies in the universality class of Lorentzian, and DT quantum gravity in the universality class of Euclidean signature. (Note that nonperturbative path integrals based on (C)DT are defined for fixed spacetime topology.) Since classical gravity in two dimensions is trivial, there are no (nontrivial) classical solutions. As a consequence, 2D quantum gravity models and the quantum geometries they generate do not possess a nontrivial classical limit and are therefore maximally ”quantum.” The curvature profiles of 2D Lorentzian and Euclidean quantum gravity will be described in sections ”CDT Quantum Gravity in D = 2” and ”DT Quantum Gravity in D = 2.”

上一节介绍的量子里奇曲率为我们提供了表征和量化量子几何 (包括远离经典性区域的量子几何) 的新工具。我们事先并不清楚 QRC 能表现出何种非平凡量子行为，但原则上没有理由认为它的复杂度会低于经典对应版本。在基于动态三角剖分的非微扰模型背景下，二维玩具模型自然成为 QRC 的首个合适试验场。二维纯引力模型的作用量由 (拓扑) 爱因斯坦-希尔伯特项和宇宙学常数项构成，根据其度规符号分为两个不同的普适类:CDT 量子引力属于洛伦兹号差普适类，DT 量子引力属于欧几里得号差普适类。(请注意，基于 (C)DT 的非微扰路径积分是针对固定时空拓扑定义的。) 由于二维经典引力是平凡的，不存在 (非平凡) 经典解。因此，二维量子引力模型及其生成的量子几何不具备非平凡经典极限，因而是最大程度“量子化”的。二维洛伦兹和欧几里得量子引力的曲率轮廓将在“D=2 下的 CDT 量子引力”和“D=2 下的 DT 量子引力”章节中介绍。

However, the main motivation for the introduction of the QRC was the need to understand the curvature properties of the emergent quantum spacetime found in full, four-dimensional quantum gravity formulated in terms of CDT [32, 33]. Its volume profile matches that of a classical de Sitter universe, with quantum volume fluctuations that likewise match a semi-classical treatment [30,34]. By investigating the curvature properties of this dynamically generated quantum universe, one would like to understand to what extent it matches the expected behavior of the classical, constantly curved de Sitter space and to quantify how its curvature deviates in a (quasi-)local sense from a perfectly homogeneous distribution. Currently available results in 4D will be summarized in section ”CDT Quantum Gravity in D = 4”.

然而，引入 QRC 的核心动机，是为了理解以 CDT 表述的完整四维量子引力中得到的演生量子时空的曲率性质 [32, 33]。该时空的体积轮廓与经典德西特宇宙一致，量子体积涨落也与半经典处理结果相符 [30,34]。通过研究这个动态生成的量子宇宙的曲率性质，我们可以了解它在多大程度上符合恒定曲率经典德西特空间的预期行为，并量化其曲率在 (准) 局域意义上与完美均匀分布的偏差。目前四维情况下已得到的结果将在“D=4 下的 CDT 量子引力”章节总结。

CDT Quantum Gravity in D = 2

D=2 下的 CDT 量子引力

The two-dimensional version of the Wick-rotated CDT path integral (10) is given by

Wick 转动后的二维 CDT 路径积分 (10) 由下式给出

$$Z = \sum_{\text{causal } T} \frac{1}{C(T)} \mathrm{e}^{-S_\lambda[T]}, \; S_\lambda[T] = \lambda N_2(T), \tag{27}$$

where the sum is taken over causal triangulations T of topology $S^1 \times S^1$, with compact spatial S^1 -universes of variable length and a compactified time direction for computational convenience. The Euclidean action $S_\lambda[T]$ consists of a cosmological-constant term, where the bare cosmological constant λ multiplies the discrete volume $N_2(T)$, counting the number of equilateral 2D triangles in T . CDT quantum gravity in 2D was first solved analytically in [18]; the continuum theory has a spectral dimension of (at most) 2 [36] and a Hausdorff dimension of 2 [18,36,37]. The fact that these dynamical dimensions happen to be equal to the topological dimension of the triangular building blocks of the regularized path integral does not imply that the quantum geometry resembles any classical geometry and even less that it is locally flat. Figure 12 shows a typical member of the ensemble of 2D CDT geometries, illustrating the large fluctuations of the spatial volume of the universe as a function of (discrete) proper time. Clearly, these universes are not individually locally flat, but it is not clear a priori what will happen to the curvature when both a manifold and an ensemble average are taken into account.

求和遍历拓扑结构 $S^1 \times S^1$ 的因果三角剖分 T，为方便计算，空间宇宙 S^1 为长度可变的紧致流形，时间方向也做紧致化处理。欧几里得作用量 $S_\lambda[T]$ 包含一项宇宙学常数项: 裸宇宙学常数 λ 乘以离散体积 $N_2(T)$，后者计数了 T 中等边二维三角形的数量。二维 CDT 量子引力最早在文献 [18] 中得到解析解；其连续理论的谱维数 (至多) 为 2[36]，豪斯多夫维数为 2[18,36,37]。这些动力学维数恰好等于正则化路径积分中三角形构造块的拓扑维数，并不意味着量子几何类似于任何经典几何，更不意味着它是局部平直的。图 12 展示了二维 CDT 几何系综中的一个典型构型，说明了宇宙空间体积随 (离散) 固有时的大幅涨落。显然，这些宇宙单个来看并非局部平直，但在同时考虑流形和系综平均的情况下，曲率会呈现何种性质，事先并不明确。

The expectation value $\langle \bar{d}_{\mathrm{av}}(\delta)/\delta \rangle$ of the curvature profile of the quantum geometry generated in 2D CDT quantum gravity was investigated in [38] for a range $N_2 \in [50k, 600k]$ of spacetime volumes, for two different time extensions $\tau = 183, 243$, and a corresponding range $\delta \in [1, \delta_{\max}]$ in terms of link distance, with $\delta_{\max} = 30, 40$, respectively. As usual, the lattice simulations were performed for fixed discrete volume N_2 , in search for a continuum (scaling) limit as $N_2 \to \infty$. Another set of measurements with $N_2 \in [100k, 250k]$ and $\tau = 183$ used the dual link distance, with $\delta_{\max} = 60$. The maximal sphere radii $\delta_{\max}$ were chosen to avoid that the results are affected significantly by topological (finite-size) effects due to the compactified time and spatial directions. Such an effect occurs whenever a shortest geodesic between two points $(q, q') \in S_p^\delta \times S_{p'}^\delta$ only exists because of the compactness and would not exist on an open patch containing the two δ -spheres. The subtle part is to control for such effects in the spatial direction, because of the strong fluctuations in the length of the spatial universe (Fig. 12). This was achieved in [38] by monitoring the relative winding numbers of geodesics between pairs (q, q') of points.

文献 [38] 针对一系列不同的时空体积 $N_2 \in [50k, 600k]$、两种不同的时间延拓 $\tau = 183, 243$，以及对应一系列不同的链路距离 $\delta \in [1, \delta_{\max}]$ (分别满足 $\delta_{\max} = 30, 40$)，研究了二维 CDT 量子引力生成的量子几何曲率轮廓的期望值 $\langle \bar{d}_{\mathrm{av}}(\delta)/\delta \rangle$。和通常一样，格点模拟在固定离散体积 N_2 下进行，以寻找 $N_2 \to \infty$ 趋近时的连续 (标度) 极限。另一组测量采用 $N_2 \in [100k, 250k]$ 和 $\tau = 183$，使用对偶链路距离，满足 $\delta_{\max} = 60$。我们选取了最大球面半径 $\delta_{\max}$，以避免结果因时间和空间方向紧致化带来的拓扑 (有限尺寸) 效应发生显著偏差。当两点 $(q, q') \in S_p^\delta \times S_{p'}^\delta$ 之间的最短短程线仅因紧致性才存在，而在包含这两个 δ 球的开片上不存在时，就会产生这类效应。由于空间宇宙长度涨落极强 (图 12)，如何在空间方向控制这类效应是这项工作的难点。文献 [38] 通过监测点对 (q, q') 之间短程线的相对绕数解决了这个问题。

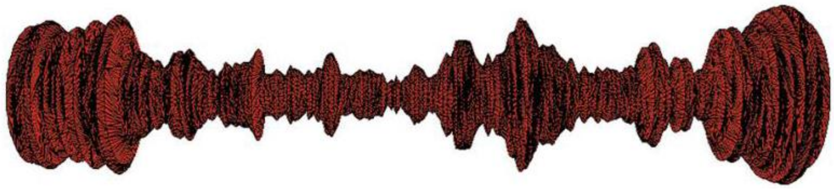


Fig. 12 Typical configuration contributing to the CDT path integral in 2D, for $N_2 = 9408$ [35], illustrating the fluctuating volume of the spatial S^1 -slices as a function of proper time (horizontal direction, cyclically

identified in simulations)

图 12 二维 CDT 路径积分中一个典型贡献构型，对应 $N_2 = 9408$ [35]，展示了空间切片 S^1 的体积涨落随固有时的变化 (水平方向为固有时，模拟中按周期性识别)

The data for the curvature profiles are shown in Fig. 13. As can be seen from the curves for the smaller volumes, their decrease for larger δ seems entirely due to finite-size effects. If one discards the data contaminated by discretization artifacts ($\delta \leq 5$) and finite-size effects (to the right of the vertical lines in the figure), they appear to fall on a common curve which increases slowly as a function of δ . It is impossible to say whether this curve will eventually asymptote to a flat curve, which would indicate an average, "effective" flatness at large coarse-graining scales; there is currently no evidence from available data that this happens. Using the dual link distance gives a similar picture. The conclusion at this stage is that the curvature profile of the 2D CDT "quantum torus," which was dubbed as "quantum-flat" in [38], does not resemble that of any classical, constantly curved geometry. As a matter of principle, this is not surprising because of the entirely nonclassical character of quantum gravity in two dimensions.

曲率分布的数据如图 13 所示。从较小体积对应的曲线可以看出，曲线在较大 δ 处下降完全是有限尺寸效应导致的。如果舍弃被离散化伪影 ($\delta \leq 5$) 和有限尺寸效应污染的数据 (即图中竖线右侧的数据)，剩余数据会落在一条随 δ 缓慢增长的公共曲线上。目前无法判断这条曲线最终是否会渐变为水平曲线——若渐变为水平曲线则表明在大粗粒化尺度上存在平均的“有效”平坦性；现有数据没有证据支持这一情况。使用对偶链路距离也会得到相似的结论。现阶段的结论是，被文献 [38] 称为“量子平坦”的二维 CDT“量子环面”的曲率分布，与任何经典常曲率几何的曲率分布都不相符。从原理上讲，这并不意外，因为二维量子引力本身就具有完全非经典的性质。

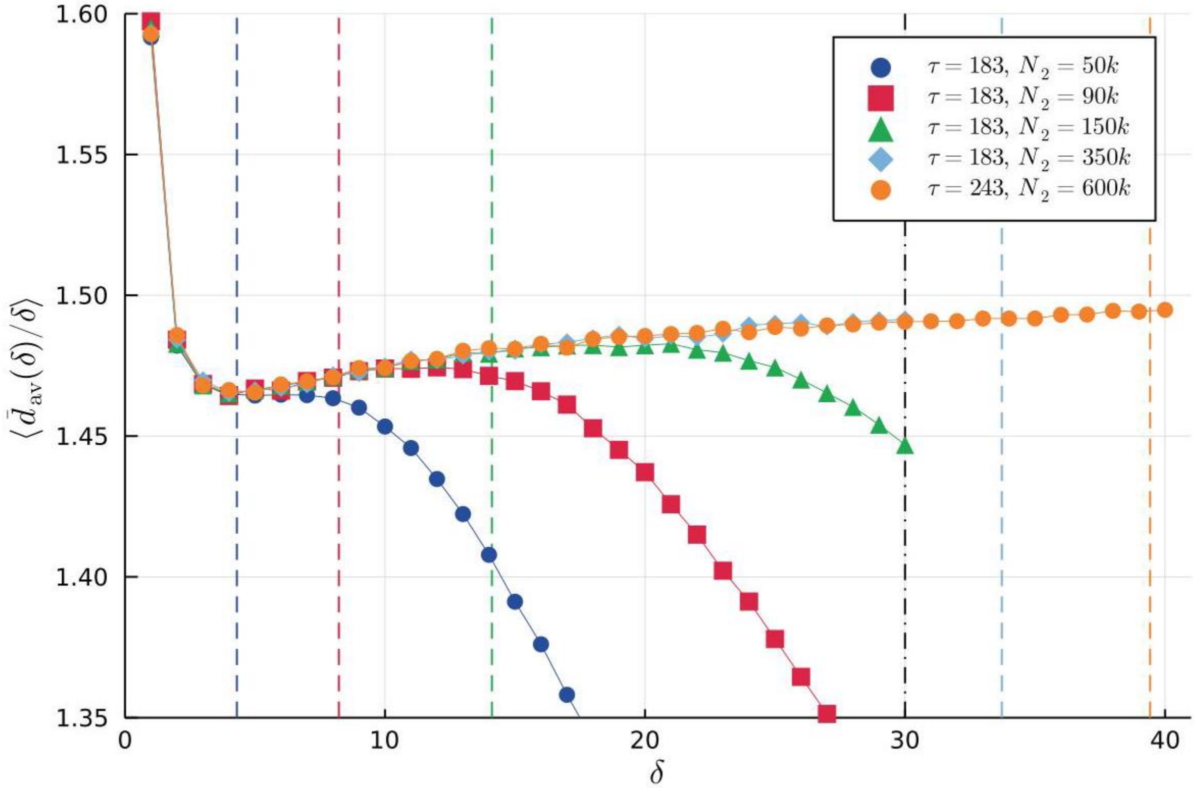


Fig. 13 Curvature profiles $\langle \bar{d}_{\mathrm{av}}(\delta)/\delta \rangle$ measured in 2D CDT quantum gravity on a torus, as a function of link distance δ , together with upper bounds on δ (vertical dashed lines), indicating the δ -ranges where finite-size effects due to spatial compactness are negligible [38]

图 13 在环面上的二维 CDT 量子引力中测得的曲率分布 $\langle \bar{d}_{\mathrm{av}}(\delta)/\delta \rangle$，是链路距离 δ 的函数，图中虚线竖线为 δ 的上界，表示空间紧致性导致的有限尺寸效应可忽略的 δ 范围 [38]

Interestingly, because of the Lorentzian nature of the CDT geometries and the associated time slicing, one can make use of the directional nature of the QRC and investigate whether it behaves differently in spacelike and timelike directions. A limited investigation in [38] looked at the two cases where the two sphere centers p, p' lie in the same spatial slice or, alternatively, have a maximally timelike separation. The curvature profile for the case of spacelike separation very much resembles that of the directionally averaged data of Fig. 13; however, the curvature profile in the timelike direction has an initial dip (for small δ) that is much less pronounced than in the other cases and for larger δ only has a very slight upward slope. Within measuring accuracy, it cannot be distinguished from a flat curve, indicating a possible anisotropy of the QRC in 2D CDT quantum gravity, which deserves further study.

有意思的是，由于 CDT 几何的洛伦兹性质和相关的时间切片，我们可以利用 QRC 的方向性，研究它在类空和类时方向的表现是否存在差异。文献 [38] 开展了有限的研究，考察了两种情况: 两个球心 p, p' 位于同一个空间切片，或是两个球心具有最大类时间隔。类空间隔情况下的曲率分布与图 13 方向平均后的数据非常相似；但类时方向的曲率分布在初始阶段 (小 δ 处) 的下凹远弱于其他情况，且在大 δ 处仅存在非常平缓的上升斜率。在测量精度范围内，它无法与水平曲线区分开，这表明二维 CDT 量子引力中的 QRC 可能存在各向异性，值得进一步研究。

DT Quantum Gravity in D = 2

D = 2 下的 DT 量子引力

We turn next to the analysis of the curvature properties of two-dimensional Euclidean quantum gravity, as captured by the (continuum limit of the) nonper-turbative DT path integral

接下来我们分析由非微扰 DT 路径积分 (连续极限下) 所描述的二维欧几里得量子引力的曲率性质

$$Z = \sum_{T} \frac{1}{C(T)} \mathrm{e}^{-S_{\lambda}[T]}, \; S_{\lambda}[T] = \lambda N_2(T). \tag{28}$$

The difference with the CDT path integral (27) is the set of two-dimensional triangulations T in the sum, which in (28) consists of all simplicial manifolds of fixed topology that can be assembled from equilateral flat triangles, and is strictly bigger than the corresponding configuration space of the Wick-rotated Lorentzian path integral (27). This difference is significant, since it leads to a different universality class of 2D quantum gravity models. More specifically, the quantum geometry of 2D Euclidean gravity [39-41], aka Liouville gravity [42], is characterized by a Hausdorff dimension of 4 and a spectral dimension of 2. Typical path integral configurations are very ”spiky” and nonclassical, as illustrated in Fig. 14. It seems that before the investigation of the QRC in [5], which looked at the standard case with S^2 -topology, no attempt had been made to associate a curvature with this quantum geometry, which in a mathematical context also goes by the name ”Brownian sphere” [41].

它与 CDT 路径积分 (27) 的区别在于求和中的二维三角剖分集合 T，(28) 中的集合包含所有可由等边平坦三角形拼接而成的固定拓扑单纯流形，严格大于维克旋转洛伦兹路径积分 (27) 对应的构型空间。这一差异十分显著，因为它使二维量子引力模型属于不同的普适类。更具体地说，二维欧几里得引力 [39-41] 又称刘维尔引力 [42]，其豪斯多夫维数为 4，谱维数为 2。典型的路径积分构型极具“尖刺状”，不属于经典构型，如图 14 所示。在文献 [5] 研究含 S^2 拓扑的标准情形之前，似乎从未有人尝试给这种量子几何赋予曲率——该量子几何在数学领域也被称为“布朗球面”[41]。

One could of course use the total deficit angle curvature, introduced in section ”Deficit Angle Curvature in Dynamical Triangulations”, but since this is fixed to a constant by the Gauss-Bonnet theorem, it does not contain any interesting new information. By contrast, although the QRC is compatible with the Gauss-Bonnet theorem on triangulations that approximate classical, two-dimensional manifolds, for $\delta \geq 2$, it does in general not satisfy the Gauss-Bonnet theorem. (Regularity conditions under which the Gauss-Bonnet theorem holds for $\delta = 1$ are examined in [43], Section 3.2.) This is a potential asset from the point of view of the non-perturbative quantum theory, since the integrated quantum Ricci scalar is not constrained to behave like that of a two-dimensional space. It is appropriate in a context where quantum geometry in a continuum limit can scale anomalously, and in particular can have an effective dimension that is different from the dimension of its elementary building blocks, as is illustrated by 2D Euclidean quantum gravity.

当然，我们可以使用“动力学三角剖分中的亏角曲率”一节引入的总亏角曲率，但根据高斯-博内定理，总亏角曲率固定为常数，无法提供任何有价值的新信息。相比之下，虽然 QRC 在逼近经典二维流形的三角剖分上满足高斯-博内定理，但对于 $\delta \geq 2$，它一般不满足高斯-博内定理。(文献 [43] 的 3.2 节考察了 $\delta = 1$ 满足高斯-博内定理的正则性条件。) 从非微扰量子理论的角度来看，这是一个潜在的优势，因为积分量子里奇标量不需要被约束为遵循二维空间的行为。这一性质适用于连续极限下量子几何存在反常标度的场景，尤其是其有效维数可以不同于基本构造单元的维数——二维欧几里得量子引力就是一个典型例子。

Fig. 14 Typical configuration in the ensemble of 2D dynamical triangulations of spherical topology

图 14 球面拓扑二维动力学三角剖分系综中的典型构型

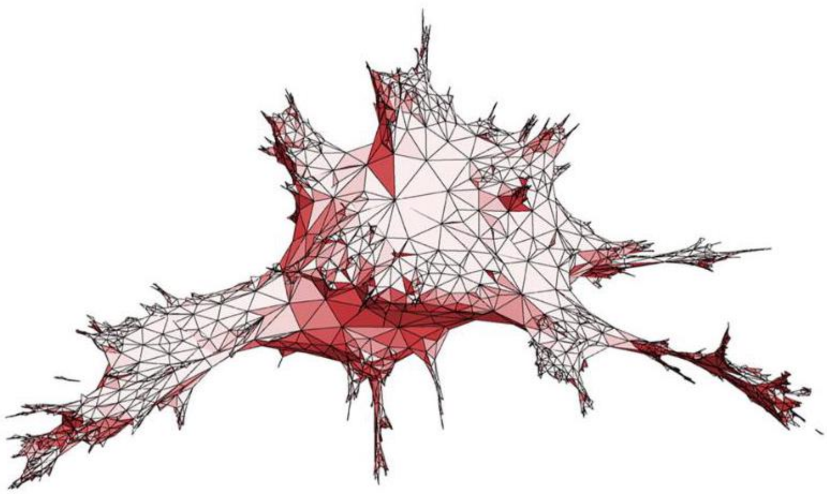
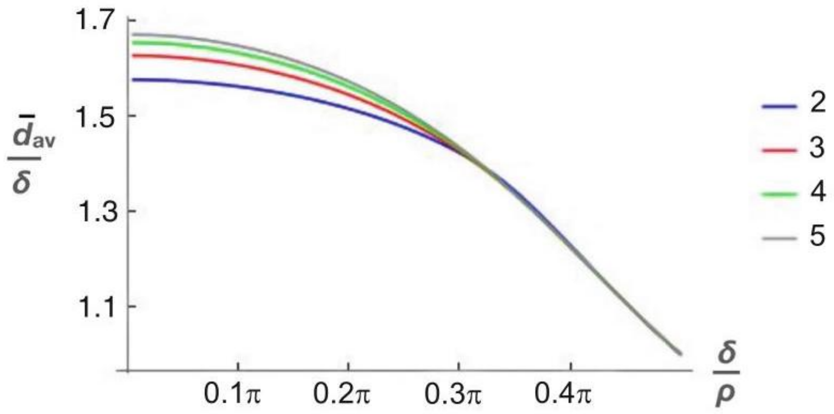


Fig. 15 Curvature profiles $\bar{d}_{\text{av}}(\delta)/\delta$ of continuum spheres in dimensions $D = 2$, 3, 4, and 5 (bottom to top), as a function of the normalized distance δ/ρ , where ρ denotes the curvature radius of the sphere [5]

图 15 连续球面的曲率轮廓 $\bar{d}_{\text{av}}(\delta)/\delta$ ，维数分别为 $D = 2$ 、3、4、5(自下而上)，是归一化距离 δ/ρ 的函数，其中 ρ 为球面的曲率半径 [5]



The expectation value $\langle \bar{d}_{\text{av}}(\delta) \rangle/\delta$ in 2D DT quantum gravity on a two-sphere was measured in Monte Carlo simulations for volumes in the range $N_2 \in [20k, 240k]$ in [5]. Since the data for link distances $\delta \gtrsim 5$ indicate a positive curvature, they were fitted to curvature profiles of constantly curved spheres in the continuum. In view of the non-canonical scaling properties of the quantum geometry, the dimension of the latter was varied in the range $D \in [2, 5]$ (Fig. 15). From fitting the measurements at a given volume N_2 to the curvature profile of a D -sphere, an effective curvature radius ρ_{eff} was extracted and used to normalize the distances δ to δ/ρ_{eff} , for better comparison with the universal curves of Fig. 15.

文献 [5] 在蒙特卡洛模拟中测量了二维球面上二维 DT 量子引力的期望值 $\langle \bar{d}_{\text{av}}(\delta) \rangle/\delta$ ，体积范围为 $N_2 \in [20k, 240k]$ 。由于链接距离 $\delta \gtrsim 5$ 的数据显示曲率为正，研究人员将数据与连续空间常曲率球面的曲率轮廓做拟合。考虑到量子几何具有非正则标度性质，拟合时将维数在 $D \in [2, 5]$ 范围内变化 (图 15)。将给定体积 N_2 下的测量结果与 D 球面的曲率轮廓拟合后，研究提取出有效曲率半径 ρ_{eff} ，并用它将距离 δ 归一化为 δ/ρ_{eff} ，以便更好地和图 15 的普适曲线对比。

Perhaps surprisingly, considering the highly nonclassical nature of the quantum geometry, a joint fit of the data for different volumes $N_2 \in [20k, 240k]$ for a fixed reference dimension D leads to a fairly good overlap with a single continuum curve for $\bar{d}_{\text{av}}/\delta$ over a whole range of δ/ρ_{eff} -values, indicating the presence of finite-size scaling. The quality of the fit improves slightly with the reference dimension and is best for $D = 5$, the case shown in Fig. 16. However, given the similarity of the curves for continuum spheres, Fig. 15, it is difficult to discriminate between the cases $D = 4$ and $D = 5$ on the basis of the sphere fits alone. One can invoke an additional selection criterion by extracting another effective dimension $\mathcal{D}$ from the scaling relation

或许出乎意料的是，考虑到量子几何高度非经典的性质，对固定参考维度 D 下不同体积 $N_2 \in [20k, 240k]$ 的数据进行联合拟合后，得到的结果在整个 δ/ρ_{eff} 取值范围内都能与 $\bar{d}_{\text{av}}/\delta$ 对应的单条连续曲线很好地重合，这表明存在有限尺寸标度。拟合质量随参考维度增大略有提升，在 $D = 5$ 时达到最优，对应图 16 所示的情况。但由于连续球体曲线的相似性 (见图 15)，仅靠球体拟合很难区分 $D = 4$ 和 $D = 5$ 这两种情况。我们可以通过标度关系提取另一个有效维度 $\mathcal{D}$ ，引入额外的选择标准

$$\rho_{\text{eff}} \propto N_2^{1/\mathcal{D}}, \tag{29}$$

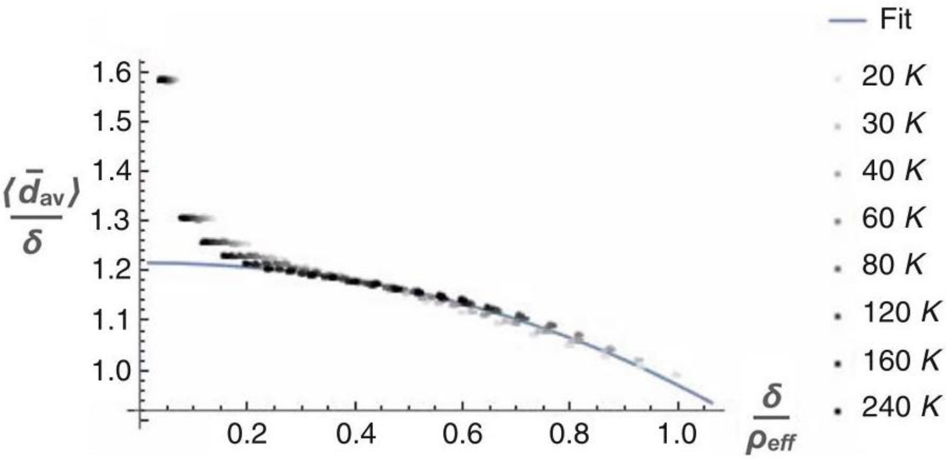


Fig. 16 Curvature profile $\langle \bar{d}_{\text{av}}(\delta)/\delta \rangle$ measured in 2D DT quantum gravity on a sphere as a function of the rescaled distance δ/ρ_{eff} , for volumes in the range $N_2 \in [20k, 240k]$, with a best fit to a 5D continuum sphere [5]

图 16 二维 DT 量子引力中球体上测得的曲率剖面 $\langle \bar{d}_{\text{av}}(\delta)/\delta \rangle$ ，它是重标度距离 δ/ρ_{eff} 的函数，体积范围为 $N_2 \in [20k, 240k]$ ，该图是对五维连续球体的最优拟合 [5]

which determines how the effective curvature radii obtained from the sphere fits at given D depend on the volume N_2 . The ansatz (29) is motivated by the fact that the curvature radius of a continuum D -sphere of two-volume V_2 behaves like $\rho \propto V_2^{1/D}$. It turns out that, depending on the dimension D and the fitting method used to obtain ρ_{eff}, $\mathcal{D}$ varies in the range $[4.8, 6.0]$, with the closest match between D and $\mathcal{D}$ obtained for $D = 5$ (restricting to integer dimensions). This led to the main conclusion of [5], namely, that the curvature profile of 2D Euclidean quantum gravity is best approximated by that of a five-dimensional continuum sphere. An analysis of the same curvature profile, including on ensembles of triangulations with weaker regularity requirements than those of simplicial manifolds, which nevertheless are known to lie in the same universality class, is currently underway (R. Loll and T. Niestadt, in preparation).

它决定了在给定 D 下通过球体拟合得到的有效曲率半径如何依赖于体积 N_2 。假设式 (29) 的动机来自以下事实: 二维体积为 V_2 的连续 D 球体，其曲率半径行为符合 $\rho \propto V_2^{1/D}$ 。结果表明，依赖于维度 D 和用于得到 $\rho_{\text{eff}}, \mathcal{D}$ 的拟合方法，相关量在 $[4.8, 6.0]$ 范围内变化，若限制为整数维度，D 与 $\mathcal{D}$ 的匹配度在 $D = 5$ 时最高。这得出了文献 [5] 的主要结论: 二维欧几里得量子引力的曲率剖面最接近五维连续球体的曲率剖面。目前，针对同一曲率剖面的更多分析正在进行中 (R. Loll 与 T. Niestadt, 待完成)，该分析涵盖了正则性要求低于单纯形流形的三角剖分系综，而这类三角剖分已知处于同一普适类。

CDT Quantum Gravity in D = 4

D=4 下的 CDT 量子引力

As already mentioned in the introduction to section ”Quantum Ricci Curvature: Quantum Applications”, a main motivation for the introduction of the QRC was the physical case of quantum gravity in four dimensions, and a closer investigation of the curvature properties of the dynamically generated de Sitter-like quantum geometry found in the 4D CDT path integral. Its Wick-rotated version was given in Eq. (10), and the explicit form of the four-dimensional action is

正如“量子里奇曲率: 量子应用”章节引言中已经提到的，引入量子里奇曲率 (QRC) 的一个主要动机是研究四维量子引力这一物理问题，以及更深入地探究四维 CDT 路径积分中得到的动力学生成类德西特量子几何的曲率性质。其威克转动形式已由式 (10) 给出，四维作用量的具体形式为

$$S[T] = -\kappa_0 N_0(T) + \Delta\left(2N_{41}(T) + N_{32}(T) - 6N_0(T)\right) + \kappa_4\left(N_{41}(T) + N_{32}(T)\right),$$

(30)

where N_{41} and N_{32} count the numbers of four-simplices of type (4,1) and $(3,2)$, respectively, with $N_{41} + N_{32} = N_4$. These two types correspond to two distinct Minkowskian simplicial building blocks before the analytic continuation to Euclidean signature and have different numbers of time- and spacelike edges. The bare coupling constants appearing in the action (30) are κ_4 , which is related to the cosmological constant, the inverse Newton constant κ_0 , and the asymmetry parameter Δ , which captures the finite relative scaling between the geodesic lengths of time- and spacelike edges and becomes a relevant coupling in the nonperturbative regime (for more details, see [2, 3]). The topology of the triangulations is $S^1 \times S^3$, with compact spatial slices of S^3 -topology and a compactified time direction for computational convenience.

其中 N_{41} 和 N_{32} 分别对类型为 (4,1) 的四维单形和 $(3,2)$ 的数量计数，且满足 $N_{41} + N_{32} = N_4$ 。解析延拓到欧几里得符号之前，这两类对应两种不同的闵氏单纯形构造块，其类时边与类空间边的数量各不相同。作用量 (30) 中的裸耦合常数包括: 与宇宙学常数相关的 κ_4 、逆牛顿常数 κ_0 ， 以及不对称参数 Δ ——该参数刻画了类时边与类空间边测地线长度之间的有限相对标度，在非微扰区域中是一个相关耦合 (更多细节见 [2, 3])。三角剖分的拓扑为 $S^1 \times S^3$ ， 为方便计算，其空间切片具有 S^3 拓扑且时间方向是紧致化的。

The Monte Carlo measurements of the curvature profile in [44] were performed at the point $(\kappa_0, \Delta) = (2.2, 0.6)$ in the de Sitter phase of 4D CDT quantum gravity, for universes with time extension $\tau = 120$,(quasi-)fixed volumes in the range $N_4 \in [150k, 1200k]$ and $\delta \leq 15$. When using the link distance, it was not possible to obtain reliable data for $\langle \bar{d}_{\rm av}(\delta)\rangle/\delta$ because of strong finite-size effects related to the presence of vertices of large order. The situation improved significantly when using the dual link distance, in the sense that one could find a nontrivial range of δ -values where the QRC data could be compared meaningfully with those of continuum spaces, at least for sufficiently large volumes N_4 . The data indicate a positive average quantum Ricci scalar, suggesting a comparison with curvature profiles for smooth spheres. Unlike in the case of 2D DT quantum gravity described in section "DT Quantum Gravity in D = 2", 4D CDT quantum gravity has a nontrivial classical limit, and both Hausdorff and spectral dimension on sufficiently large scales are compatible with the classically expected value of 4 [32,33,45]. This suggests a comparison with the classical curvature profiles of spheres of dimension $D = 4$.

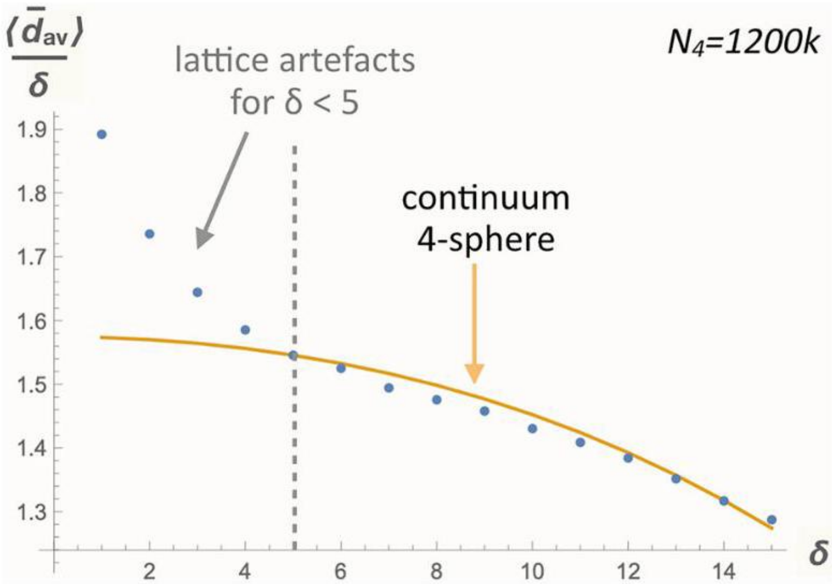
> 文献 [44] 中对曲率分布的蒙特卡洛测量是在四维 CDT 量子引力德西特相的点 $(\kappa_0, \Delta) = (2.2, 0.6)$ 处进行的，被测宇宙的时间延拓为 $\tau = 120$ ，(准) 固定体积处于 $N_4 \in [150k, 1200k]$ 到 $\delta \leq 15$ 范围内。使用链路距离时，由于大阶顶点存在带来的强有限尺寸效应，无法得到 $\langle \bar{d}_{\rm av}(\delta)\rangle/\delta$ 的可靠数据。使用对偶链路距离后情况得到显著改善: 我们可以找到 δ 值的一个非平凡范围，在该范围内 QRC 数据可以和连续空间的数据进行有意义的对比，至少对于足够大的体积 N_4 是如此。数据表明平均量子里奇标量为正，说明可以和光滑球面的曲率分布做对比。不同于“D=2 下的 DT 量子引力”章节描述的二维 DT 量子引力，四维 CDT 量子引力具有非平凡经典极限，足够大尺度下的豪斯多夫维数和谱维数都与经典预期值 4 相容 [32,33,45]，这说明可以将其与维数为 $D = 4$ 的球面的经典曲率分布做对比。

Like in the curvature measurements in 2D quantum gravity, the fitting must take into account an (a priori unknown) shift in the measurement data, corresponding to the non-universal parameter $c_{\rm av}$ in relation (25), and data points for $\delta < 5$ were not considered because of discretization artifacts. (One has the choice between an additive and a multiplicative shift, which has little effect on the final result. Here, following [5,43], an additive shift was chosen.) For the smallest investigated four-volumes, the quality of the fits of the 4D data to continuum curves is only moderate. However, for increasing N_4 , the curve through the measurement data in the range $\delta \in [5, 15]$ becomes gradually more convex, characteristic of the behavior of a positively curved continuum sphere of radius ρ . This trend is clearly visible in the CDT data and mirrors the behavior of the 2D DT system with increasing volume N_2 , but the approach to spherical behavior is slower than in the 2D DT case. The quality of the sphere fit at the largest volume $N_4 = 1200k$ is decent (Fig. 17), but not as good as the corresponding fit at the largest volume $N_2 = 240k$ for the 2D DT measurements. This is not particularly surprising since one would expect on general grounds that a four-dimensional system shows a slower convergence as a function of the total volume than a two-dimensional one, even if by some measures two-dimensional quantum gravity has an effective dimensionality larger than two. A further consistency check was performed in [44] by measuring the volumes of three-dimensional spherical shells and showing that the extracted curvature radius ρ broadly agrees with that obtained from the QRC measurements.

与二维量子引力中的曲率测量类似，拟合必须考虑测量数据中 (先验未知的) 偏移，该偏移对应关系式 (25) 中的非普适参数 c_{av}，且由于离散化伪影，未纳入 $\delta < 5$ 的数据点。(可在加法偏移和乘法偏移中选择，二者对最终结果影响很小；本文遵循文献 [5,43] 选择加法偏移。) 对于所研究的最小四体积，四维数据对连续曲线的拟合质量仅为中等。然而，随着 N_4 增大，区间 $\delta \in [5, 15]$ 内测量数据对应的曲线逐渐变得更凸，这是半径为 ρ 的正曲率连续球面的典型行为。该趋势在 CDT 数据中清晰可见，反映了二维 DT 系统随体积 N_2 增大的行为，但趋近球面行为的速度比二维 DT 情况更慢。最大体积 $N_4 = 1200k$ 下的球面拟合质量尚可 (图 17)，但不如二维 DT 测量中最大体积 $N_2 = 240k$ 对应的拟合结果好。这并不特别出人意料，因为一般而言，人们会预期四维系统作为总体积的函数，收敛速度慢于二维系统，即便从某些衡量标准来看二维量子引力的有效维数大于二维。文献 [44] 通过测量三维球壳的体积做了进一步一致性检验，结果表明提取出的曲率半径 ρ 与 QRC 测量得到的结果大体一致。

Fig. 17 Curvature profile $\langle \bar{d}_{av}(\delta)/\delta \rangle$ measured in 4D CDT quantum gravity on $S^1 \times S^3$ in the de Sitter phase, as a function of the dual link distance δ , for volume $N_4 = 1200k$, together with a best fit to the curvature profile of a continuum four-sphere [44]

图 17 在德西特相的 $S^1 \times S^3$ 上、四维 CDT 量子引力中测量得到的曲率剖面 $\langle \bar{d}_{av}(\delta)/\delta \rangle$，它是体积 $N_4 = 1200k$ 下对偶链接距离 δ 的函数，同时给出了其对连续四维球面曲率剖面的最佳拟合 [44]



To summarize, there is strong evidence from the data collected in the window $\delta \in [5, 15]$ that the expectation value of the average quantum Ricci scalar is compatible with that of a classical four-sphere. Given that the physical length scale at which the curvature is probed is not more than about 10 Planck lengths [44], this is quite a remarkable result, and a further piece of evidence for the de Sitter nature of the dynamically generated quantum geometry in 4D CDT quantum gravity. In addition, analogous to what was described for 2D CDT in section ”CDT Quantum Gravity in D = 2”, the directional character of the QRC was employed to compare the behavior of the curvature profile in a maximally timelike and maximally spacelike direction, adapted to the formulation on the dual lattice. The result of this preliminary study, unlike that in two dimensions, found no significant difference between the time- and spacelike measurements, other than a constant shift, corresponding to different constants c_{av} associated with the anisotropy of the underlying lattice structure, which was already noted for flat lattices in section ”Construction and Implementation”. It suggests a restoration of local ”rotational” symmetry, which is present on a classical de Sitter space.

总而言之，区间 $\delta \in [5, 15]$ 内收集的数据有充分证据表明，平均量子里奇标量的期望值与经典四维球面的期望值相容。鉴于曲率探测的物理长度尺度不超过约 10 个普朗克长度 [44]，这是一项相当了不起的结果，也进一步证明了四维 CDT 量子引力中动力学生成的量子几何具有德西特性质。此外，类似于“D=2 的 CDT 量子引力”一节中介绍的二维 CDT 的情况，我们利用 QRC 的方向特性，比较了适配于对偶格点表述的最大类时方向和最大类空方向上曲率剖面的行为。这项初步研究的结果发现，与二维情况不同，类时和类空测量之间除了常数偏移外没有显著差异，该常数偏移对应于底层格点结构各向异性相关的不同常数 c_{av}，这一点在“构造与实现”一节的平坦格点中已经提到。这表明局域“转动”对称性得到恢复，而该对称性是经典德西特空间所具备的。

Summary and Outlook

总结与展望

As we have seen in sections ”Quantum Ricci Curvature” and ”Quantum Ricci Curvature: Quantum Applications”, the quantum Ricci curvature is a well-defined notion of Ricci curvature, which has been devised for application in nonperturbative quantum gravity, where typical spacetime configurations contributing to the path integral are neither classical nor smooth. The QRC K_q at scale δ of Eq. (20) is defined operationally in terms of distance and volume measurements, which is a typical feature of (ingredients of) observables in a nonperturbative regime. It provides a benchmark for the type of curvature information one is able to access in such a regime, which is dictated not only by the highly nonclassical character of the geometry but also by the requirement of diffeomorphism invariance and background independence, typically implying some form of integration or averaging. The message here is that natural, well-defined quantum operators are not of the form of individual quantized components $\widehat{R}^{\kappa}_{\lambda\mu\nu}(x)$ of the classical Riemann tensor, but are nonlocal, composite operators like the QRC.

正如我们在“量子里奇曲率”和“量子里奇曲率: 量子应用”两节中所见，量子里奇曲率是定义明确的里奇曲率概念，专为非微扰量子引力应用设计，在该理论中，对路径积分有贡献的典型时空构型既不是经典的，也不是光滑的。式 (20) 中标度 δ 下的 QRC K_q 通过距离和体积测量以操作方式定义，这是非微扰区域观测量 (的组成部分) 的典型特征。它为该区域可获取的曲率信息类型提供了基准，这种可及性不仅由几何的高度非经典性质决定，还受微分同胚不变性与背景独立性的要求限制，这些要求通常意味着某种形式的积分或平均。本文的核心结论是: 自然且定义明确的量子算子并非经典黎曼张量单个量子化分量 $\widehat{R}^{\kappa}_{\lambda\mu\nu}(x)$ 的形式，而是像 QRC 这样的非局域复合算子。

The QRC is a ”tensorial” quantity in the sense that it encodes information beyond the Ricci scalar, depending on $\mathrm{Ric}(v, v)$ and its covariant derivatives, as is illustrated by the expansions (21) for the average sphere distance on Riemannian spaces for infinitesimal δ . This direction dependence has so far only been used to a limited extent in quantum applications, namely, in CDT models to get a first idea of the dependence of the QRC on time- vs. spacelike directions, as mentioned in sections ”CDT Quantum Gravity in D = 2” and ”CDT Quantum Gravity in D = 4”. (In a classical context, it has been used to investigate the anisotropic curvature properties of a two-dimensional ellipsoid (see [43], and work with G. Clemente, to appear).) This feature will become more prominent in upcoming studies of the curvature properties of coupled gravity-matter systems, including point particles.

QRC 是“张量”量级，因为它编码了超出里奇标量的信息，它依赖于 Ric (v, v) 及其协变导数，无穷小 δ 下黎曼空间平均球面距离的展开式 (21) 就说明了这一点。迄今为止，这种方向依赖性在量子应用中仅得到有限利用，正如“D=2 的 CDT 量子引力”和“D=4 的 CDT 量子引力”两节提到的，它只在 CDT 模型中被用于初步了解 QRC 对类时方向和类空方向的依赖关系。(在经典语境下，它已被用于研究二维椭球的各向异性曲率性质，见文献 [43]，以及与 G. Clemente 合作的待发表工作。) 这一特性将在未来研究耦合引力-物质系统 (包括点粒子) 的曲率性质中发挥更重要的作用。

The simplest diffeomorphism-invariant observable constructed from the QRC is the curvature profile (25), which only depends on the Ricci scalar, at least at lowest nontrivial order in δ . (Since the standard prescription for the QRC for $\varepsilon = \delta$ (cf. section ”Construction and Implementation”) is not rotationally symmetric, it still has a directional character; however, choosing $\delta = 0$ is associated with full rotational symmetry.) Importantly, although the curvature profile is an integrated, global (albeit scale-dependent) observable, its application in full 4D CDT quantum gravity has already led to two major new results. Firstly, it implies a finite, renormalized average curvature on higher-dimensional systems of dynamical triangulations, unlike the deficit angle curvature of section ”Deficit Angle Curvature in Dynamical Triangulations”. Secondly, it has produced additional evidence that the classical limit of the quantum geometry found in this candidate theory of quantum gravity is indeed a de Sitter space.

由 QRC 构造的最简单的微分同胚不变观测量是曲率轮廓 (25)，它至少在 δ 的最低非平凡阶仅依赖于里奇标量。(由于 $\varepsilon = \delta$ 对应的 QRC 标准方案 (参见“构造与实现”一节) 不具有旋转对称性，它仍保留方向特性；不过，选择 $\delta = 0$ 即可得到完全旋转对称性。) 重要的是，尽管曲率轮廓是积分得到的整体 (尽管依赖标度) 观测量，它在完整四维 CDT 量子引力中的应用已经带来了两项重要新成果。首先，和“动力三角剖分中的亏缺角曲率”一节介绍的亏缺角曲率不同，它给出了高维动力三角剖分系统上有限的重整化平均曲率。其次，它提供了额外证据，证明在这个量子引力候选理论中，量子几何的经典极限确实是德西特空间。

Overall, the advent of a well-defined quasi-local notion of curvature in nonpertur-bative quantum gravity implies a major advance, which is likely to be transferable to approaches different from CDT, if they have suitable computational means to implement the QRC. Its computational implementation and measurement, especially in higher dimensions, is fairly complex, and any further numerical optimization will be welcome. The QRC offers a plethora of possibilities for constructing new observables, since the only (quasi-)local physical quantities available to date were the Hausdorff and spectral dimensions. Quantities involving local vertex orders (numbers of d -simplices meeting at a vertex), closely related to deficit angles, have also been used on occasion, but their relation to any physical, renormalized notion of curvature is tenuous.

总体而言，在非微扰量子引力中引入定义明确的准局域曲率概念是一项重大进展，如果其他非 CDT 方法具备合适的计算手段来实现 QRC，这一概念很可能也适用于它们。QRC 的计算实现与测量 (尤其在高维中) 相当复杂，任何进一步的数值优化都是受欢迎的。QRC 为构造新观测量提供了丰富可能，因为迄今为止可用的 (准) 局域物理量只有豪斯多夫维数和谱维数。涉及顶点局域阶 (即汇聚于一个顶点的 d 单形数量) 的量与亏缺角密切相关，有时也会被使用，但它们和任何物理的、重整化的曲率概念的关联都十分薄弱。

Examples of new observables where the local QRC is expected to play a prominent role are homogeneity measures of the type introduced in [14] and two-point functions (R. Loll and J. van der Duin, in preparation), studied earlier in the context of Euclidean dynamical triangulations [46,47]. Determining the degree of homogeneity and isotropy of the de Sitter-like quantum spacetime will be an essential step in trying to relate it to continuum descriptions of the very early universe, where these symmetries are usually assumed to be present. The behavior of two-point functions captures essential spacetime properties of any quantum field theory, while two-point functions of spatial slices of a cosmological background are closely related to the power spectrum [48]. In other words, we are interested both in spacetime curvature and in the spatial curvature of slices of constant time or of other subspaces. (The curvature profile of two-dimensional spatial slices in three-dimensional CDT quantum gravity was investigated in [49, 50].) Many interesting applications remain to be explored, given the many invariant aspects of classical general relativity that are expressed in terms of curvature and the fact that we have only just begun to understand the true quantum properties of curvature in a Planckian regime.

预计局域量子里奇曲率 (QRC) 将发挥重要作用的新可观测物实例包括文献 [14] 提出的这类均匀性测度，以及早前在欧几里得动力三角化背景下研究过的两点关联函数 (R. Loll 与 J. van der Duin，待发表)[46,47]。确定类德西特量子时空的均匀性与各向同性程度，是将其与极早期宇宙连续描述关联起来的关键一步——极早期宇宙模型中通常默认存在这些对称性。两点关联函数的行为能刻画任意量子场论的核心时空性质，而宇宙背景空间切片的两点关联函数与功率谱密切相关 [48]。换言之，我们既关注时空曲率，也关注等时切片或其他子空间的空间曲率。(三维因果动态三角化 (CDT) 量子引力中二维空间切片的曲率分布已在文献 [49,50] 中研究。) 经典广义相对论中诸多用曲率表述的不变性内容，加之我们才刚刚开始理解普朗克能区曲率的真实量子性质，因此仍有大量有趣的应用有待探索。

Acknowledgments I am grateful to numerous individuals for discussion and collaboration on various aspects of quantum curvature, in particular, N. Klitgaard, J. Brunekreef, G. Clemente, J. Ambjørn, T. Niestadt, A. Silva, and J. van der Duin. This research was supported in part by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through the Department of Innovation, Science and Economic Development and by the Province of Ontario through the Ministry of Colleges and Universities.

致谢我非常感谢众多研究者就量子曲率的各方面内容与我开展讨论与合作，尤其要感谢 N. Klitgaard、J. Brunekreef、G. Clemente、J. Ambjørn、T. Niestadt、A. Silva 和 J. van der Duin。本研究部分得到圆周理论物理研究所支持。圆周理论物理研究所的研究得到加拿大政府通过创新、科学与经济发展部，以及安大略省通过学院与大学部的资助。

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